# Solitary Wave Calculations for Erosion Strength

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**Abstract**

For the calculation of steep water waves a Lagrangian method is presented which makes ist easier to derive and understand the complicated nonlinear structure of the equations of motion (EOM), for the behaviour of water surfaces. In addition it is shown, that in this formulation it is possible to model an EOM which describes water waves in the more general case of varying depth.

# 1 Introduction

Water waves with larger amplitudes, both in deep and in shallow water, have pretty different properties in contrast to the well known low amplitude sinoidal waves.

The properties of steep water waves, travelling into shallower regions of the shore, may be of interest for the marine biology due to their impact on e. g. water flow turbulence, sea bottom erosion, chemical exchange with the atmosphere or momentum and energy transfer. We present a way of calculating their properties.

The behaviour of water waves could be calculated from Hydrodynamics *if* the exact equation of motion would be known. This however, is the problem. Thus various model assumptions had been proposed and respective equations of motion set up. These have however a complicated mathematical structure. This made it difficult to understand the basic properties in terms of the equation of motion (EOM). In addition, most solutions have been gained by linearizing (and thus simplifying) the EOM. It turns out, however, that the larger amplitude waves have nonlinear properties, which are inadvertantly partially skipped this way.

We propose to start the derivation of an EOM by a Lagrangian function of the transverse surface field. The design of Lagrangians is known to be suitable for the case of complicated and coupled motions in Abstract Quantum Field Theory. The advantage is, that each field enters with additive kinetic and potential terms, and that the coupling between different fields as well as the nonlinear selfcoupling enter by relatively simple additive terms. This greatly helps the design of a Lagrangian suited to a given set of phenomena.

As an example, assume a water basin with the surface elevation $\eta(z)$. Then the Lagrangian has to have a kinetic term $\frac{1}{(2m)}(\partial\eta)^2$, with the moment of inertia $m$, and a potential (or mass term), $(\eta)^2$ both of which serve to a EOM for the wellknown free or linear waves. If this surface Lagrangian is modelled to have in addition higher order terms, such as $(\eta)^n$, $n = 3, 4, 5..$ these turn out to be responsible for selfcoupling nonlinear effects. One of which is the wellstudied solitary wave or *soliton*.

Right away, because of the higher powers of the Lagrangian terms, one sees, that these solitons can occur only if the amplitudes $\eta$ are sufficiently high, otherwise an $\eta^3$ term would be small against the terms of the free waves and thus would not contribute to the resultant EOM and its final solution. Sometimes the soliton regime of the EOM is called *weak nonlinear*, because one still has as solutions waves, with however different properties, and not just turbulence and foam.

In the case of more than one field, e.g. the mentioned surface waves and now in addition, e.g. a longitudinal sound or other longitudinal disturbance $\chi$, the Lagrangian method has the advantage to easily cope with it, by just introducing the respective kinetic and potential term for the new field *plus* a coupling term between these two fields, by adding a product of the type $\eta^2\chi$. This procedure has proven its validity in many fields of coupled motions, and could even be applied to the coupling of ocean sea quakes with the ocean surface waves, called *Tsunami*.

In this contribution we design and apply the Lagrangian method to the case of ocean waves in deep water, as coupled to itself and as second application surface waves in shallow water, that is, coupled to the depths of the water column.

We will set up the respective Lagrangian, derive the resulting EOM and discuss the properties of the solutions in comparison to linear waves.

We will give some arguments that the strength of erosion due to the waves in shallow water that is the horizontal velocity of the water at the bottom, can be calculated and comes out to be much different and pretty larger than for linear waves of the same size.

# 2 Properties of Solitary Waves

Conventional low amplitude *sinoidal* water surface waves do have the wellknown properties

- The amplitude is independent of its wavelength;
- The velocity depends on the wavelength (shorter waves travel more slowly). Wave packages disperse, diassemble with time .

In contrast, solitons show as the respective properties

- The amplitude depends on the velocity (or other parameters) of the wave and vice versa;
- No disperion: a solitary wave is stable in it's form with time.

Thus, a soliton is a local, stable but propagating object.

We demonstrate the advantage of the Lagrangian Method, proposed here, by two applications, for shallow and then for deep water waves.

# 3 shallow water waves

## 3.1 The equation of motion

The equation of motion for shallow water waves have been extensively discussed in the literature[1, 2]. Their complicated intricate nonlinear structure is exemplified by the wellstudied *Korteweg-de Vries* equation, a differential equation for the wave amplitude $\eta$ as a function of the water depth $h_0$ travelling along $x$ with time $t$. The constant $c_0$ gives the coupling between the depth and $\eta$, the strength of which governed by the gravitation constant $g$, $c_0 = \sqrt{gh_0}$. This equation holds only for constant water depth, $h_0 = const.$

$$\frac{\partial \eta}{\partial t} - c_0\left(1 + \frac{3}{2}\frac{\eta}{h_0}\right)\frac{\partial \eta}{\partial x} + \frac{1}{6}c_0 h_0^2 \frac{\partial^3 \eta}{\partial x^3} = 0 \ .$$

The solutions are

$$\eta = \eta_o \mathrm{sech}^2\left\{\left(\frac{3\eta_o}{4h_o^3}\right)^{\frac{1}{2}}(x - vt)\right\} \ . \tag{1}$$

These Korteweg-deVries solutions are one of the oldest case of a soliton, with all its properties as mentioned above. Specifically, the velocity is given by

$$v = c_0\left(1 + \frac{1}{2}\frac{\eta_0}{h_0}\right) \ . \tag{2}$$

that is, the higher the soliton, the faster it is.

Although the KdV equation does give interesting soliton solutions, the EOM is, at least to us, by no means vivid, and thus no feeling may be developed to design a respective EOM for more complicated cases such as rapidly changing depths.

## 3.2 The Lagrangian

For a linear sinoidal wave the Lagrangian for the surface amplitude $\eta$ has to have a kinetic and a 'free' or so called 'mass' term. It is

$$L = \frac{1}{2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \alpha_1 \eta^2 \ . \tag{3}$$

The simplest nontrivial Lagrangian has thus to have at least one additional term. It is an equation with e. g. a cubic term in $\eta$. This is

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \alpha_2 \eta^2 + \beta_2 \eta^3 \ , \tag{4}$$

which leads, by the proper derivatives directly to the nonlinear Korteweg-deVries equation of motion. In contrast to its intricate mathematical structure the Lagrangian consists only of additive terms, with the cubic in $\eta$ one known to be responsible for soliton wave type solutions.

# 4 Steep water waves

## 4.1 The Lagrangian

Thus we apply this method of constructing a Lagrangian and from there calculating the respective equation of motion to the non-linear Schrödinger equation as well. The respective Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\eta}{\partial x}\right)^2 + \alpha_3\eta^2 + \beta_3\eta^4 \ . \tag{5}$$

The quartic term is known to be responsible for a 'self-coupling' of waves.

## 4.2 Generalization

While the Korteweg-deVries case is for coupling to the shallow sea-bottom, the nonlinear Schrödinger is valid for deep water self-coupling of surface waves.

The generalization for steep water waves for any water depths including shallow water may be impossible to find starting directly from the equations of motion. For the Lagrangian method it is extremely easy: we just add the two nonlinear terms to

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\eta}{\partial x}\right)^2 + \alpha\eta^2 + \beta\eta^3 + \gamma\eta^4 \ . \tag{6}$$

The coefficients have now to be designed thus that the two limiting cases are encluded.

$\alpha$ has to go continuously from $\alpha_2$ to $\alpha_3$ , as a function of depth, and for small amplitudes to $\alpha_1$. From first principles they should be about equal.

$\beta$ should be set up to depend on the inverse of the depth.

Finally, $\gamma$ has to become zero for shallow water, a suitable ansatz may thus be $a/(1 + \exp^{(\eta - h_0)/b})$ with a coupling strength $a$ and a coupling 'width' $b$.

At the moment we evaluate these parameters to optimally be applicable to sea water waves to travel from deep to shallow water.

# 5 Examples

In figures 1, 2, and 3 we show some examples of specific calculations.

A deep water soliton solution is shown in figure 1. It is just a group of three waves locally concentrated in space in both surface directions.

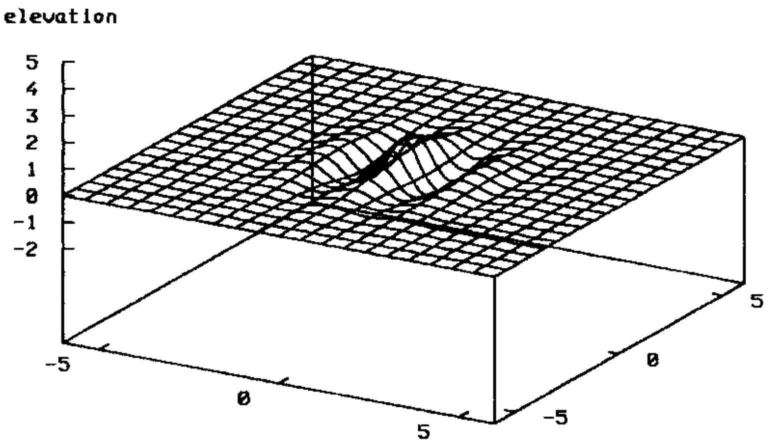


Figure 1: Deep water soliton

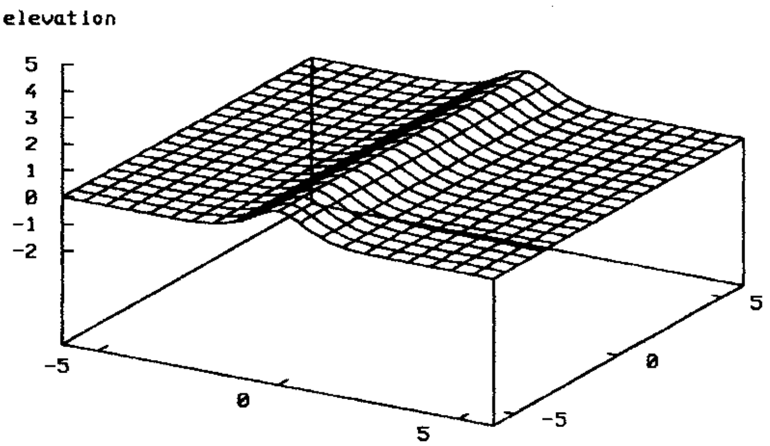


Figure 2: Shallow water soliton

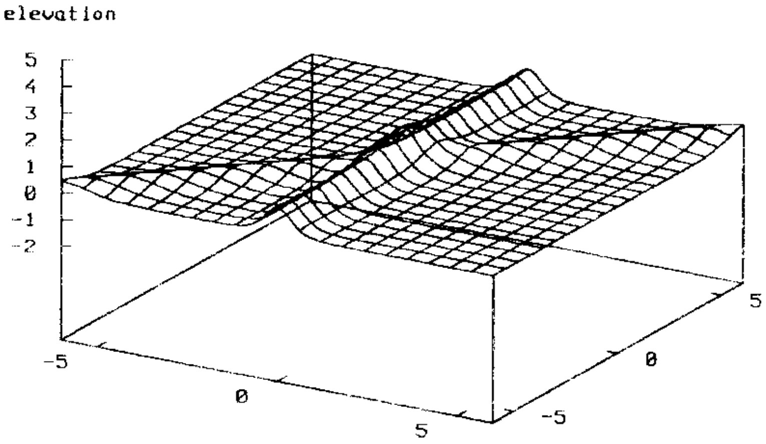


Figure 3: Calculation for two crossing shallow water solitons

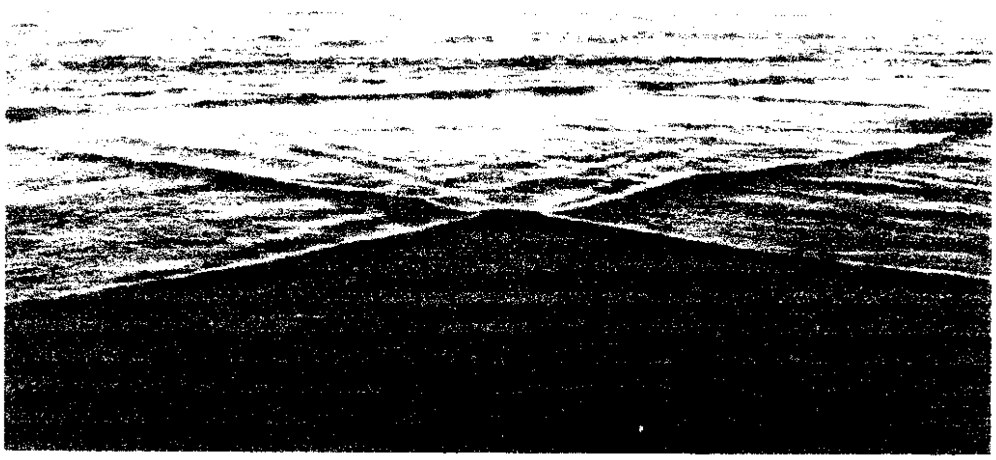

Figure 4: Interaction of two shallow water waves

Travelling into shallower water the front- and after-waves diminish to feed the main central wave. The wave length perpendicular to the travelling direction increases, see figure 2.

In figure 3 we give a special example for the case of two crossing shallow water solitons to be compared to the photograph of such a case, given in figure 4 as taken from [2].

# 6 Summary

A new method has been presented to design equations of motion for steep water waves travelling in deep or shallow water. Their solutions gives their properties denied as input to marine biology scenarios, both close to the sea surface and in shallow water.

# 7 Acknowledgements

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# References

[1] G. B. Whitham. *Linear and Nonlinear Waves.* John Wiley & Sons, New York, Chichester, Brisbane, Toronto, Singapore, 1974.

[2] Mark J. Ablowitz and Harvey Segur. *Solitons and the Inverse Scattering Transform.* Siam, Philadelphia, 1981.