Boson-fermion and baryon mapping: Construction of collective subspaces

III. Application of the baryon mapping to many-baryon states

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Abstract

The theory developed in Part I of this series is applied to the general case of a system consisting of colourfree quark triplets, whose quantum numbers are chosen as collective trifermion indices. The appropriate mapping technique to be used here is the baryon mapping. It is demonstrated that the original multiquark states can be exactly represented by states of colourfree ideal baryons. Besides, collective extended images are derived for a class of fermionic operators leaving the collective fermion space invariant.

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I. Introduction

In the first paper of this series,[1] hereafter referred to as Part I, the applicability of the boson-fermion and the baryon mapping[2,3] in the case of a truncated set of three-fermion quantum numbers has been studied. In the formalism developed there, only certain values of these quantum numbers, the collective indices, are allowed to appear in the states of the different many-particle spaces involved in the mapping. The images of the fermionic operators have to be modified accordingly, which leads to the concept of collective extended images. The existence of such an image could be proved for any linear fermionic operator leaving the collective subspace of the original multifermion space invariant. The physical results valid in this subspace are then exactly reproduced in the model space, a subspace of the collective ideal space.

In the second paper of this series,[4] in the following referred to as Part II, the theory described in Part I has been applied to the single-orbit quark shell model[5,6] developed by Petry and coworkers. There, the basic fermions are quarks, and the collective three-quark systems considered in Part II possess the quantum numbers of nucleons. In the present article, the more general case of baryons, i. e., colourfree quark triplets, will be treated. The appropriate mapping technique for this purpose is the baryon mapping. The treatment presented here is not restricted to a special physical model but is kept sufficiently general to apply to a whole class of Hamiltonians leaving the collective fermion space invariant.

The paper has the following organization: In Sec. II, collective trifermion indices are defined according to the scheme of Part I. The subject of Sec. III is the construction of the model space, and the derivation of collective extended image operators is described in Sec. IV. Section V contains a short summary of the results and conclusions. Throughout the paper, extensive reference is made to Parts I and II, whose equations will be quoted by their number preceded by I or II, respectively. In addition, some results presented in Ref. 2 are used.

II. Definitions

The notation used here is essentially analogous to that of Part II. The creation operator of a quark possessing m = 3M states is written in the form

$$a^{+}_{i\mu} \qquad \text{with} \quad i = 1,2,3 \quad \text{and} \quad \mu = 1,\dots,M \ , \tag{2.1}$$

where the latin index i denotes the colour state, and the greek index μ stands for the remaining quantum numbers.

In accordance with the convention of Part I, Sec. III.B, the collective triquark annihilation and creation operators are defined to be the *colourfree* operators

$$(aaa)_{\alpha\beta\gamma} := 1/6 \sum_{abc} \sum_{\rho\sigma\tau} C^{\alpha\beta\gamma *}_{a\rho b\sigma c\tau} \; a_{c\tau} a_{b\sigma} a_{a\rho} \tag{2.2}$$

and

$$(a^{+}a^{+}a^{+})_{\alpha\beta\gamma} := (aaa)^{+}_{\alpha\beta\gamma} \tag{2.3}$$

(see Eqs. (I.3.22) - (I.3.23)). Here, the collective trifermion index is a triplet $(\alpha\beta\gamma)$. The coupling coefficients

$$C^{\alpha\beta\gamma}_{a\rho b\sigma c\tau} := 1/\sqrt{6} \; \varepsilon_{abc} \; \delta^{+}_{(\rho\sigma\tau)(\alpha\beta\gamma)} \tag{2.4}$$

with the totally antisymmetric Kronecker tensor ε_{abc} (cf. the colourfree triquark operator introduced in Refs. 5 - 8) and the symbol

$$\delta^{+}_{(\rho\sigma\tau)(\alpha\beta\gamma)} := \delta_{\rho\alpha}\delta_{\sigma\beta}\delta_{\tau\gamma} + \delta_{\sigma\alpha}\delta_{\tau\beta}\delta_{\rho\gamma} + \delta_{\tau\alpha}\delta_{\rho\beta}\delta_{\sigma\gamma} + \delta_{\sigma\alpha}\delta_{\rho\beta}\delta_{\tau\gamma} + \delta_{\tau\alpha}\delta_{\sigma\beta}\delta_{\rho\gamma} + \delta_{\rho\alpha}\delta_{\tau\beta}\delta_{\sigma\gamma} \tag{2.5}$$

are completely antisymmetrized in their indices a, b, c and totally symmetric both in α, β, γ and in ρ, σ, τ. (The superscript + of $\delta^{+}_{(\rho\sigma\tau)(\alpha\beta\gamma)}$ indicates the symmetrization.) The collective trifermion operators are therefore independent of the order of their index triplets. The normalization of the coefficients is chosen in such a way that the orthogonality relation

$$1/6 \sum_{rst} \sum_{\rho\sigma\tau} C^{\alpha\beta\gamma *}_{r\rho s\sigma t\tau}\, C^{\varkappa\lambda\mu}_{r\rho s\sigma t\tau} = \delta^{+}_{(\alpha\beta\gamma)(\varkappa\lambda\mu)} \tag{2.6}$$

(cf. Eq. (I.3.24)) is satisfied. As a consequence of Eqs. (2.4) and (2.5), the collective triquark operator (2.2) obtains the simple form

$$(aaa)_{\alpha\beta\gamma} = 1/\sqrt{6} \sum_{abc} \varepsilon_{abc}\, a_{c\gamma} a_{b\beta} a_{a\alpha} \; . \tag{2.7}$$

Application of the baryon mapping to the operator $(aaa)_{\alpha\beta\gamma}$ leads to the extended image (I.3.31), which is here given by

$$\big((aaa)_{\alpha\beta\gamma}\big)_{\mathcal{J}} = b_{\alpha\beta\gamma} \tag{2.8}$$

with the collective ideal baryon annihilation operator

$$b_{\alpha\beta\gamma} := 1/6 \sum_{abc} \sum_{\rho\sigma\tau} C^{\alpha\beta\gamma *}_{a\rho b\sigma c\tau}\, b_{a\rho b\sigma c\tau} \tag{2.9}$$

$$= 1/\sqrt{6} \sum_{abc} \varepsilon_{abc}\, b_{a\alpha b\beta c\gamma} \tag{2.10}$$

of Eq. (I.3.26). The colourfree operator $b_{\alpha\beta\gamma}$ defined in this way is completely symmetric in its indices α, β, γ. (In spite of the similar appearance, it must not be confused with the totally *anti*symmetric ideal baryon operator introduced in Sec. II of Part I, where the greek indices have a different meaning.) The anticommutation rules of the colourfree ideal baryon operators read (cf. Eqs. (I.3.28)):

$$\left.\begin{aligned} \{b_{\alpha\beta\gamma}, b^+_{\varkappa\lambda\mu}\} &= \delta^+_{(\alpha\beta\gamma)(\varkappa\lambda\mu)} \, , \\ \{b_{\alpha\beta\gamma}, b_{\varkappa\lambda\mu}\} &= \{b^+_{\alpha\beta\gamma}, b^+_{\varkappa\lambda\mu}\} = 0 \, . \end{aligned}\right\} \tag{2.11}$$

In the following, the abbreviation

$$(a^+a^+a^+)_{\alpha\beta\gamma} =: T^+_{\alpha\beta\gamma} \tag{2.12}$$

known from Part I will be used. In the case treated here, the collective fermion space $\mathcal{F}_C$ introduced in Sec. IV.B of Part I is spanned by the states of the form

$$T^+_{\rho_n\sigma_n\tau_n} \dots T^+_{\rho_1\sigma_1\tau_1}|0\rangle \tag{2.13}$$

with $\rho_1, \sigma_1, \tau_1, \dots, \rho_n, \sigma_n, \tau_n = 1,\dots,M$ and all possible integer values of n. The collective ideal space $\mathcal{J}_C$ is the Fock space of the colourfree ideal baryons and consists of all linear combinations of the states with the structure

$$(T_{\rho_n\sigma_n\tau_n})^+_{\mathcal{J}} \dots (T_{\rho_1\sigma_1\tau_1})^+_{\mathcal{J}} \, |0) = b^+_{\rho_n\sigma_n\tau_n} \dots b^+_{\rho_1\sigma_1\tau_1}|0) \, , \tag{2.14}$$

where, again, arbitrary values of $\rho_1, \sigma_1, \tau_1, \dots, \rho_n, \sigma_n, \tau_n$, and n may occur.

III. The model space

As shown in Part I, Sec. V.B, the model space $\mathcal{J}_{CPC}$ defined in Sec. IV.B of Part I is spanned by the states of the form (I.5.18),

$$\hat{C}\ (T^{+}_{\rho_n\sigma_n\tau_n})_{\mathcal{P}}\ \dots\ (T^{+}_{\rho_1\sigma_1\tau_1})_{\mathcal{P}}\ |0)$$

$$= (T^{+}_{\rho_n\sigma_n\tau_n})'_{\mathcal{J}}\ \dots\ (T^{+}_{\rho_1\sigma_1\tau_1})'_{\mathcal{J}}\ |0)\ , \tag{3.1}$$

with all possible values of ρ_1, σ_1, τ_1, ..., ρ_n, σ_n, τ_n, and n. These states are the collective images of the multiquark states (2.13). In Eq. (3.1), $\hat{C}$ is the orthogonal projection operator onto the collective ideal space $\mathcal{J}_C$, $(T^{+}_{\rho\sigma\tau})_{\mathcal{P}}$ is the exact image and $(T^{+}_{\rho\sigma\tau})'_{\mathcal{J}}$ a collective extended image of the operator $T^{+}_{\rho\sigma\tau}$.

The collective extended image operator $(T^{+}_{\rho\sigma\tau})'_{\mathcal{J}}$ can be constructed according to an algorithm analogous to that described in Sec. IV.B of Part II. The procedure begins with the expression

$$\hat{C}\,(T^{+}_{\rho\sigma\tau})_{\mathcal{P}}\,\hat{P}_{C}$$

$$= \hat{C}\,1/\sqrt{6}\,\sum_{rst}\varepsilon_{rst}\,(a^{+}_{r\rho}a^{+}_{s\sigma}a^{+}_{t\tau})_{\mathcal{P}}\,\hat{P}_{C} \tag{3.2}$$

$$\begin{aligned} = \hat{C}\,1/\sqrt{6}\,\sum_{rst}\varepsilon_{rst}\Big(& b^{+}_{r\rho s\sigma t\tau} \\ & + 1/2\sum_{kl\iota}\sum_{\varkappa\lambda\mu}\big(b^{+}_{k\varkappa l\lambda r\rho}b^{+}_{s\sigma t\tau l\mu} + b^{+}_{k\varkappa l\lambda s\sigma}b^{+}_{t\tau r\rho l\mu} + b^{+}_{k\varkappa l\lambda t\tau}b^{+}_{r\rho s\sigma l\mu}\big)\, b_{k\varkappa l\lambda l\mu} \\ & + 1/8\sum_{\substack{klx\\ \iota jy}}\sum_{\substack{\varkappa\mu\xi\\ \lambda\nu\eta}} b^{+}_{k\varkappa l\lambda r\rho}b^{+}_{l\mu j\nu s\sigma}b^{+}_{x\xi y\eta t\tau}\, b_{k\varkappa l\mu x\xi}b_{l\lambda j\nu y\eta}\Big)\,\hat{P}_{C}\,, \end{aligned} \tag{3.3}$$

which is derived by means of Eqs. (2.7) and (I.3.30). As in Part I, Sec. IV.B, the symbol $\hat{P}_C$ stands for the orthogonal projection operator onto the physical collective subspace $\mathcal{P}_C$, the image of $\mathcal{F}_C$ in the ideal space. (In the last line of Eq. (3.3), the projector $\hat{P}$ onto the physical subspace $\mathcal{P}$ has been omitted from the exact image (I.3.30) because $\hat{P}\ \hat{P}_C = \hat{P}_C$, as shown in Eq. (I.4.14).)

In the next step, the creation operators on the right side of (3.3) are reduced to their collective components. As a consequence of Eq. (I.3.27), the collective part of an operator $b^{+}_{r\rho s\sigma t\tau}$ is given by

$$(b^{+}_{r\rho s\sigma t\tau})' := 1/6\sum_{\alpha\beta\gamma} C^{\alpha\beta\gamma *}_{r\rho s\sigma t\tau}\, b^{+}_{\alpha\beta\gamma} \tag{3.4}$$

$$= 1/\sqrt{6}\,\varepsilon_{rst}\, b^{+}_{\rho\sigma\tau}\,. \tag{3.5}$$

(The factor 1/6 appearing in Eq. (3.4) results from the special normalization used here.)

For further simplification of the operator $\hat{C}\ (T^{+}_{\rho\sigma\tau})_{\mathcal{P}}\ \hat{P}_C$, several properties of the Kronecker tensor ε_{rst} are needed. From the equation

$$\sum_r \varepsilon_{rkl}\varepsilon_{rij} = \delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li} \ , \tag{3.6}$$

which is easily verified, one deduces

$$\sum_{rs} \varepsilon_{rsl}\varepsilon_{rsj} = 2\ \delta_{lj} \tag{3.7}$$

and finally

$$\sum_{rst} \varepsilon_{rst}\varepsilon_{rst} = 6 \ . \tag{3.8}$$

Another important relation following from (3.6) is

$$\begin{aligned}
&\sum_{rst} \varepsilon_{rst}\varepsilon_{klr}\varepsilon_{ijs}\varepsilon_{xyt} \\
&= 1/3 \Big(\sum_{rs} \varepsilon_{klr}\varepsilon_{ijs} \Big(\sum_t \varepsilon_{rst}\varepsilon_{xyt}\Big) \\
&\qquad + \sum_{rt} \varepsilon_{klr}\varepsilon_{xyt} \Big(\sum_s \varepsilon_{rst}\varepsilon_{ijs}\Big) \\
&\qquad + \sum_{st} \varepsilon_{ijs}\varepsilon_{xyt} \Big(\sum_r \varepsilon_{rst}\varepsilon_{klr}\Big)\Big) \\
&= 1/3 \big(\varepsilon_{klx}\varepsilon_{ijy} - \varepsilon_{kly}\varepsilon_{ijx} \\
&\qquad + \varepsilon_{klj}\varepsilon_{xyi} - \varepsilon_{kli}\varepsilon_{xyj} \\
&\qquad + \varepsilon_{ijk}\varepsilon_{xyl} - \varepsilon_{ijl}\varepsilon_{xyk}\big) \ .
\end{aligned} \tag{3.9}$$

(Equivalent but simpler expressions are possible, too, but the comparatively complicated formula (3.9) ensures that the resulting variant of $(T^+_{\rho\sigma\tau})'_{\mathcal{T}}$ is completely symmetric in ρ, σ, τ.)

Following the same instructions as given in Sec. IV.B of Part II for the case of the nucleon creation operator, one finally arrives at the collective extended image operator

$$
\begin{aligned}
&(T^+_{\rho\sigma\tau})'_{\mathcal{T}} \\
&= b^+_{\rho\sigma\tau} + 1/6 \sum_{\kappa\lambda\mu} (b^+_{\kappa\lambda\rho} b^+_{\sigma\tau\mu} + b^+_{\kappa\lambda\sigma} b^+_{\tau\rho\mu} + b^+_{\kappa\lambda\tau} b^+_{\rho\sigma\mu})\, b_{\kappa\lambda\mu} \\
&\qquad + 1/72 \sum_{\substack{\kappa\mu\xi \\ \lambda\nu\eta}} \big(b^+_{\kappa\lambda\rho} b^+_{\mu\nu\sigma} b^+_{\xi\eta\tau} \\
&\qquad\qquad + b^+_{\kappa\lambda\sigma} b^+_{\mu\nu\tau} b^+_{\xi\eta\rho} \\
&\qquad\qquad + b^+_{\kappa\lambda\tau} b^+_{\mu\nu\rho} b^+_{\xi\eta\sigma} \big)\, b_{\mu\nu\xi} b_{\kappa\lambda\eta} \, .
\end{aligned}
\tag{3.10}
$$

(Here, the last sum has been written in a form where the symmetry in ρ, σ, τ becomes immediately visible.)

The operator $(T^+_{\rho\sigma\tau})'_{\mathcal{T}}$ is difficult to handle because of its complex structure. For an explicit construction of the collective image states (3.1), the representation (I.5.15), given by

$$
\begin{aligned}
&\hat{C}\, (T^+_{\rho_n\sigma_n\tau_n})_{\mathcal{P}} \cdots (T^+_{\rho_1\sigma_1\tau_1})_{\mathcal{P}}\, |0) \\
&= \hat{C}\, (T^+_{\rho_n\sigma_n\tau_n})_{\mathcal{T}} \cdots (T^+_{\rho_1\sigma_1\tau_1})_{\mathcal{T}}\, |0) \, ,
\end{aligned}
\tag{3.11}
$$

is better suited. According to Eqs. (5.21a) and (5.17) of Ref. 2, the states

$$(T^+_{\rho_n\sigma_n\tau_n})_{\mathcal{J}} \ \dots \ (T^+_{\rho_1\sigma_1\tau_1})_{\mathcal{J}} \ |0)$$

$$= (1/6)^{n/2} \sum_{(rst)_n} \varepsilon_{r_n s_n t_n} \dots \varepsilon_{r_1 s_1 t_1} \; (a^+_{r_n\rho_n} a^+_{s_n\sigma_n} a^+_{t_n\tau_n})_{\mathcal{J}} \ \dots \ (a^+_{r_1\rho_1} a^+_{s_1\sigma_1} a^+_{t_1\tau_1})_{\mathcal{J}} \ |0) \qquad (3.12)$$

(see Eq. (2.7)) can be written as

$$(T^+_{\rho_n\sigma_n\tau_n})_{\mathcal{J}} \ \dots \ (T^+_{\rho_1\sigma_1\tau_1})_{\mathcal{J}} \ |0)$$

$$= (1/6)^{n/2} \sum_{(rst)_n} \varepsilon_{r_n s_n t_n} \dots \varepsilon_{r_1 s_1 t_1}$$

$$\sum_{[P']} \mathrm{sign}(P') \; P'(b^+_{r_n\rho_n s_n\sigma_n t_n\tau_n} \dots b^+_{r_1\rho_1 s_1\sigma_1 t_1\tau_1})|0) \ . \qquad (3.13)$$

The notation used here has been adopted from Ref. 2. The symbol $(rst)_n$ stands for the 3n indices r_1, s_1, t_1, ..., r_n, s_n, t_n of the first summation. P' is a permutation acting on the 3n *double* indices $(r_1\rho_1)$, $(s_1\sigma_1)$, $(t_1\tau_1)$, ..., $(r_n\rho_n)$, $(s_n\sigma_n)$, $(t_n\tau_n)$, and sign(P') takes the value +1 if P' is even and −1 if P' is odd. As described in Sec. V.A of Ref. 2, permutations that would produce equal terms in Eq. (3.13) are considered as equivalent. Consequently, the set of all possible permutations of 3n indices falls into equivalence classes, and each class [P'] is represented in (3.13) by an arbitrary element P'.

When the projection operator $\hat{C}$ is applied to Eq. (3.13), the ideal baryon operators $b^+_{r\rho s\sigma t\tau}$ are reduced to their collective components (3.5). As a result, the collective image states (3.11) take on the form

$$\hat{C}\ (T^{+}_{\rho_n\sigma_n\tau_n})_{\mathcal{J}}\ \ldots\ (T^{+}_{\rho_1\sigma_1\tau_1})_{\mathcal{J}}\ |0)$$

$$= (1/6)^n \sum_{[P']} \mathrm{sign}(P') \sum_{(rst)_n} \varepsilon_{r_n s_n t_n} \ldots \varepsilon_{r_1 s_1 t_1}\ P'(\varepsilon_{r_n s_n t_n} \ldots \varepsilon_{r_1 s_1 t_1})\ P'(b^{+}_{\rho_n\sigma_n\tau_n} \ldots b^{+}_{\rho_1\sigma_1\tau_1})|0)\ . \qquad (3.14)$$

Now, the first permutation operator P' acts only on the latin indices r_1, s_1, t_1, ..., r_n, s_n, t_n, whereas the second one produces a synchronous interchange of the greek indices ρ_1, σ_1, τ_1, ..., ρ_n, σ_n, τ_n.

For the special case of *two* ideal baryons, the factor consisting of the Kronecker tensors is easily calculated by means of Eqs. (3.7) and (3.8), and the collective image states (3.14) are found to be

$$\hat{C}\ (T^{+}_{\alpha\beta\gamma})_{\mathcal{J}}\ (T^{+}_{\rho\sigma\tau})_{\mathcal{J}}\ |0)$$

$$\begin{aligned} = \quad & b^{+}_{\alpha\beta\gamma} b^{+}_{\rho\sigma\tau}|0) \\ & - 1/3\ (\ b^{+}_{\tau\beta\gamma} b^{+}_{\rho\sigma\alpha} + b^{+}_{\alpha\tau\gamma} b^{+}_{\rho\sigma\beta} + b^{+}_{\alpha\beta\tau} b^{+}_{\rho\sigma\gamma} \\ & \qquad + b^{+}_{\sigma\beta\gamma} b^{+}_{\rho\alpha\tau} + b^{+}_{\alpha\sigma\gamma} b^{+}_{\rho\beta\tau} + b^{+}_{\alpha\beta\sigma} b^{+}_{\rho\gamma\tau} \\ & \qquad + b^{+}_{\rho\beta\gamma} b^{+}_{\alpha\sigma\tau} + b^{+}_{\alpha\rho\gamma} b^{+}_{\beta\sigma\tau} + b^{+}_{\alpha\beta\rho} b^{+}_{\gamma\sigma\tau})|0)\ . \end{aligned} \qquad (3.15)$$

In the case of the quark shell model,[5,6] it should be noted that a reduction of $\mathcal{J}_{CPC}$ to the subspace of collective multi*nucleon* image states does *not* lead to the model space constructed in Part II, where the collective triquark quantum numbers were those of nucleons (cf. Eq. (II.4.22)). For two nucleons represented by operators with the structure

$$N_\gamma^+ := \sqrt{6} \sum_{\alpha\beta} g_{\alpha\beta}\ T^+_{\alpha\beta\gamma} \tag{3.16}$$

of Eq. (II.2.11), the formula (3.15) yields the result

$$\hat{C}\ (N_\gamma^+)_{\mathcal{J}}\ (N_\tau^+)_{\mathcal{J}}\ |0)$$

$$= 4/3\ (N_\gamma)^+_{\mathcal{J}}\ (N_\tau)^+_{\mathcal{J}}\ |0) - 8 \sum_{\substack{\alpha\beta\\ \rho\sigma}} g_{\alpha\beta} g_{\rho\sigma}\ b^+_{\rho\beta\gamma} b^+_{\alpha\sigma\tau}|0)\ , \tag{3.17}$$

whereas in the formalism of Part II the corresponding state is given by

$$\frac{4\Omega+2}{3\Omega+3}\ (N_\gamma)^+_{\mathcal{J}}\ (N_\tau)^+_{\mathcal{J}}\ |0) \tag{3.18}$$

with the constant Ω = M/2 (see Eqs. (II.4.22) and (II.3.17)).

IV. Collective extended image operators

The operators considered in this section are composed of terms with the structure

$$A_{\alpha\beta} := \sum_{i} a^{+}_{i\alpha} a_{i\beta} \tag{4.1}$$

and belong to the simplest nontrivial linear operators mapping $\mathcal{F}_C$ into itself. The baryon image of $A_{\alpha\beta}$ is derived from Eq. (5.28) of Ref. 2 and reads:

$$(A_{\alpha\beta})_{\mathcal{P}} = 1/2 \sum_{kli} \sum_{\varkappa\lambda} b^{+}_{k\varkappa l\lambda i\alpha} b_{k\varkappa l\lambda i\beta} \hat{P} \ . \tag{4.2}$$

Starting from the expression $\hat{C}\ (A_{\alpha\beta})_{\mathcal{P}}\ \hat{P}_C$ and using the same method as in Sec. IV.B of Part II, one is led to the collective extended image

$$(A_{\alpha\beta})'_{\mathcal{J}} = 1/2 \sum_{\varkappa\lambda} b^{+}_{\varkappa\lambda\alpha} b_{\varkappa\lambda\beta} \ . \tag{4.3}$$

For all fermionic operators that can be expanded into products of factors with the structure $A_{\alpha\beta}$, collective extended images are easily constructed from the building blocks (4.3). A Hamiltonian of the general form

$$\hat{H} := \sum_{i} \sum_{\alpha\beta} h_{\alpha\beta}\ a^{+}_{i\alpha} a_{i\beta} + \sum_{ij} \sum_{\substack{\alpha\beta \\ \gamma\delta}} w_{\alpha\beta\gamma\delta}\ a^{+}_{i\alpha} a^{+}_{j\beta} a_{i\gamma} a_{j\delta} \ , \tag{4.4}$$

e. g., can be rewritten as

$$\hat{H} = \sum_{i} \sum_{\alpha\beta} \Big(h_{\alpha\beta} + \sum_{\mu} w_{\alpha\mu\mu\beta} \Big)\ a^{+}_{i\alpha} a_{i\beta}$$
$$- \sum_{ij} \sum_{\substack{\alpha\beta \\ \gamma\delta}} w_{\alpha\beta\gamma\delta}\ a^{+}_{i\alpha} a_{i\gamma} a^{+}_{j\beta} a_{j\delta} \ , \tag{4.5}$$

and a representation in terms of colourfree ideal baryons is given by

$$\hat{H}'_{\mathcal{J}} = 1/2 \sum_{\substack{\alpha\beta \\ \varkappa\lambda}} h_{\alpha\beta}\, b^{+}_{\varkappa\lambda\alpha} b_{\varkappa\lambda\beta} - \sum_{\substack{\alpha\beta\mu \\ \gamma\delta}} w_{\alpha\beta\gamma\delta}\, b^{+}_{\alpha\beta\mu} b_{\gamma\delta\mu}$$

$$+ 1/4 \sum_{\substack{\varkappa\lambda\alpha \\ \mu\nu\beta \\ \gamma\delta}} w_{\alpha\beta\gamma\delta}\, b^{+}_{\varkappa\lambda\alpha} b^{+}_{\mu\nu\beta} b_{\varkappa\lambda\gamma} b_{\mu\nu\delta} \;. \tag{4.6}$$

(Here, the result obtained by means of Eq. (4.3) has been rearranged in normal order.)

An example is the pairing force operator (II.2.9) of the quark shell model,[6]

$$\hat{H} = G/4 \sum_{\substack{ijr \\ kl}} \sum_{\substack{\alpha\beta \\ \gamma\delta}} \varepsilon_{ijr} \varepsilon_{klr}\, g_{\alpha\beta} g_{\gamma\delta}\, a^{+}_{i\alpha} a^{+}_{j\beta} a_{k\gamma} a_{l\delta} \;, \tag{4.7}$$

which is expressed in the more transparent form

$$\hat{H} = G/2 \sum_{ij} \sum_{\substack{\alpha\beta \\ \gamma\delta}} g_{\alpha\beta} g_{\gamma\delta}\, a^{+}_{i\alpha} a^{+}_{j\beta} a_{i\gamma} a_{j\delta} \tag{4.8}$$

with the aid of Eq. (3.6). The collective extended image (4.6) of $\hat{H}$ is found to be

$$\hat{H}'_{\mathcal{J}} = -\,G/2 \sum_{\substack{\alpha\beta\mu \\ \gamma\delta}} g_{\alpha\beta} g_{\gamma\delta}\, b^{+}_{\alpha\beta\mu} b_{\gamma\delta\mu}$$

$$+ G/8 \sum_{\substack{\varkappa\lambda\alpha \\ \mu\nu\beta \\ \gamma\delta}} g_{\alpha\beta} g_{\gamma\delta}\, b^{+}_{\varkappa\lambda\alpha} b^{+}_{\mu\nu\beta} b_{\varkappa\lambda\gamma} b_{\mu\nu\delta} \;. \tag{4.9}$$

When applied to the collective image vectors of the lowest and the first excited states in the model, this operator reproduces the eigenvalues[6]

$$E := -G\,(n\,(2\Omega+3-n) - l\,(2\Omega+3-l)) \tag{4.10}$$

of the pairing force operator. The corresponding eigenstates contain n-l nucleons and l Δ-particles, where l takes the values 0 or 1. The Δ-particle is represented in the original multiquark space by an operator of the form[6]

$$D_d^+ := \sqrt{6} \sum_{\alpha\beta\gamma} d_{\alpha\beta\gamma}\, T^+_{\alpha\beta\gamma} \tag{4.11}$$

with a totally symmetric tensor $d_{\alpha\beta\gamma}$ possessing the property

$$\sum_{\alpha\beta} d_{\alpha\beta\gamma}\, g_{\alpha\beta} = 0 \,. \tag{4.12}$$

If the collective extended image operator $\hat{H}'_{\mathcal{J}}$ of the pairing force as given by Eq. (4.9) is truncated to its purely nucleonic components, the result does *not* agree with the collective extended image (II.5.5) constructed in Part II, since, as shown in Sec. III, the respective model spaces of the multinucleon states are not identical.

V. Summary and conclusions

In this paper, the baryon mapping has been applied to the general case of many-baryon systems. It has been demonstrated that the states of any system consisting of colour-singlet triquarks can be exactly modeled by states of colourfree ideal baryons. In this way, a relation between real and ideal baryons is established. Moreover, a general solution for the construction of collective extended images has been presented for a class of fermionic operators which leave the collective fermion space invariant. Since, as proved by Petry and coworkers,[7,8] *any* colourfree 3n-quark state can be written as a linear combination of states with n colourfree quark triplets, the results given above are valid for *all* colour-singlet states of 3n quarks.

Future fields of application for the formalism presented here and in the two preceding papers could possibly be the study of multibaryonic properties, the calculation of hypernuclear spectra, and the development of new models for the interaction between baryons.

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