# Boson-fermion and baryon mapping: Construction of collective subspaces

## II. Example: A simple quark shell model

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## Abstract

The formalism derived for the boson-fermion and the baryon mapping in Part I of this series is applied to the single-orbit quark shell model developed by Petry and coworkers for the description of nuclear properties. The multiquark space is reduced to the collective subspace of multinucleon states. A collective representation of states and operators in terms of bosons and ideal fermions or in terms of ideal baryons, respectively, is constructed. The results of the original model in the multinucleon space are exactly reproduced. Finally, the two transformations are compared with a related mapping technique recently published by Pittel, Engel, Dukelsky, and Ring.

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# I. Introduction

In the first paper of this series,[1] hereafter referred to as Part I, a theory has been developed for the application of the boson-fermion and the baryon mapping[2,3] in the case of a restriction to a subset of three-fermion quantum numbers, the so-called collective indices. In this formalism, the different many-particle spaces involved in the mapping are replaced by truncated spaces where only collective indices are allowed. For any linear fermionic operator leaving the collective subspace of the multifermion space invariant, a collective extended image operator could be proved to exist. It acts on states of the model space, where an exact representation of the original physical results is possible.

The purpose of the present paper is to illustrate this theory by a practical example. The two mapping techniques are applied to the single-orbit quark shell model constructed by Petry et al.[4,5] for the microscopic description of nuclear properties. The elementary constituents of this model are quarks, and the nucleus is treated as a multiquark system in a spherical bag. The interaction of the quarks has the structure of a pairing force acting only between quarks which possess the same excitation energy and whose total angular momenta and isospins are coupled to zero. This approach leads to a quark-diquark structure of the nucleon, where the diquark is in a state of vanishing total angular momentum and isospin and the third quark acts as a valence quark carrying the respective quantum numbers of the nucleon. In the present article, a system of nucleons is considered, and the nucleonic quantum numbers are chosen as collective indices.

The quark shell model has already served as an example for another kind of mapping technique recently proposed by Pittel, Engel, Dukelsky, and Ring.[6] Similar to the boson-fermion and the baryon mapping, this method is an extension of the well-known Dyson boson mapping[7,8] to multiquark systems with three-quark clustering. The formalism employed by Pittel et al. is motivated by the quark-diquark model of the baryon and is based on an iterative procedure. In the first step, the conventional Dyson boson mapping for an odd number of fermions[8] is applied, leading to a representation of the original three-quark subsystems in terms of boson-fermion pairs. Here, the boson carries the quantum numbers of the diquark, whereas the fermion symbolizes the unpaired third quark. A second mapping transforms the resulting boson-fermion systems into new particles which behave like fermions with three-quark quantum numbers and are therefore termed triplet fermions. Contrary to the ideal baryons of the baryon mapping,[2,3] they are not totally antisymmetrized in their three single-quark indices but reflect the symmetry properties of the boson-fermion pairs. In the present paper, the results obtained by Pittel et al. for nucleons in the single-orbit version of the quark shell model[6] are contrasted with the results of the boson-fermion and the baryon mapping, and the characteristic features of the different approaches are pointed out.

The plan of the paper is as follows: In Sec. II, the basic definitions of the single-orbit quark shell model developed by the Petry group are summarized. For details, the reader is referred to the original literature[4,5,9,10] on the model and its extensions. The next three sections are devoted to the boson-fermion and the baryon representation of the model and follow the approach presented in Part I, whose equations will be quoted here by their number preceded by I. The collective ideal space $\mathcal{I}_C$ and the model space $\mathcal{I}_{CPC}$ defined in Part I are introduced in Secs. III and IV, respectively, and Sec. V deals with the construction of a collective extended image of the pairing force operator. In a final discussion in Sec. VI, the mathematical formalism underlying the alternative method of Pittel et al. is analyzed and compared with the procedure described in the preceding sections.

## II. The mathematical problem

In the single-orbit version of the quark shell model,[4,5] a valence shell is considered where the quantum numbers specifying the radial excitation and the total and orbital angular momentum of a quark have fixed values. A quark creation operator is then written as

$$a^{+}_{i\mu} \qquad \text{with} \quad i = 1,2,3 \quad \text{and} \quad \mu = 1,\ldots,2\Omega \ . \tag{2.1}$$

Here, the latin index i stands for the colour state, and the quantum numbers $m_\mu$ and $\tau_\mu$ representing the z components of total angular momentum and isospin, respectively, are included in the greek index $\mu$. As in Ref. 6, the total number of values $m_\mu$ is denoted by $\Omega$. Since two quark flavours, u and d, are needed in the model, the number of different quark states $(i\mu)$ is

$$m := 6\Omega \ . \tag{2.2}$$

As mentioned in the Introduction, a nucleon in the model is composed of a diquark and a single quark. The diquark consists of two quarks whose isospins and total angular momenta are coupled to zero and is therefore in a colour-antitriplet state. The notation used by Petry and coworkers is chosen in such a way that the three possible diquark states are characterized by ordinary colour indices. The creation operator of a diquark in the state r can then be defined as

$$A^{+}_{r} := \sum_{ab} \sum_{\alpha\beta} \varepsilon_{abr} \, g_{\alpha\beta} \, a^{+}_{a\alpha} a^{+}_{b\beta} \tag{2.3}$$

(cf. Refs. 4 - 5 and 9 - 10). Here, $\varepsilon_{abr}$ is the usual totally antisymmetric

Kronecker tensor, and $g_{\alpha\beta}$ denotes the coupling coefficient of angular momentum and isospin:[5]

$$g_{\alpha\beta} := (-1)^{\Omega/2-m_\alpha-\tau_\alpha} \, \delta_{m_\alpha,-m_\beta} \, \delta_{\tau_\alpha,-\tau_\beta} \, . \tag{2.4}$$

It possesses the properties

$$g_{\alpha\beta} = g_{\beta\alpha} \, , \tag{2.5}$$

$$\sum_{\alpha} g_{\alpha\beta} g_{\alpha\gamma} = \delta_{\beta\gamma} \, , \tag{2.6}$$

$$\sum_{\alpha\beta} g_{\alpha\beta} g_{\alpha\beta} = 2\Omega \, . \tag{2.7}$$

The operator of the pairing interaction between the quarks in the valence shell has the structure[5]

$$\hat{H} := - G/4 \sum_{r} A_r^+ A_r \tag{2.8}$$

$$= G/4 \sum_{\substack{abr \\ cd}} \sum_{\substack{\alpha\beta \\ \gamma\delta}} \varepsilon_{abr} \varepsilon_{cdr} \, g_{\alpha\beta} g_{\gamma\delta} \, a_{a\alpha}^{+} a_{b\beta}^{+} a_{c\gamma} a_{d\delta} \tag{2.9}$$

with the coupling constant G. For a colourfree system of 3n quarks in the shell, the eigenspace of $\hat{H}$ belonging to lowest energy is spanned by the states[4,5]

$$\left.\begin{aligned} & N_{\rho_n}^{+} \ldots N_{\rho_1}^{+} |0\rangle \\ & \text{with} \quad \rho_1,\ldots,\rho_n = 1,\ldots,2\Omega \, , \\ & \qquad \rho_i \neq \rho_j \quad \text{for} \quad i,\, j = 1,\ldots,n \quad \text{and} \quad i \neq j \, . \end{aligned}\right\} \tag{2.10}$$

Here, the creation operator

$$N^{+}_{\rho} := \sum_{abr} \sum_{\alpha\beta} \varepsilon_{abr}\, g_{\alpha\beta}\, a^{+}_{a\alpha} a^{+}_{b\beta} a^{+}_{r\rho} \tag{2.11}$$

$$= \sum_{r} A^{+}_{r} a^{+}_{r\rho} \tag{2.12}$$

describes a nucleon of the valence shell in the state $\rho$, while the vacuum state $|0\rangle$ of the single-orbit model in general represents a core of filled shells. The eigenvalue of $\hat{H}$ for a state with n nucleons in the outer shell is[5]

$$E := -\, Gn\, (2\Omega+3-n)\ . \tag{2.13}$$

In the following, the space spanned by the multinucleon states (2.10) with $n = 0,1,\ldots,2\Omega$ is chosen as the collective fermion space $\mathcal{F}_C$ introduced in Part I, Sec. IV.B. Accordingly, the collective triquark creation operators in the sense of Part I are, up to a normalization factor, given by $N^{+}_{\rho}$, and the collective diquark operators have the structure of $A^{+}_{r}$, again up to a multiplicative constant.

## III. The collective ideal space

In order to construct the collective ideal space $\mathcal{I}_C$, one proceeds as in Secs. III and IV of Part I.

### A. Boson-fermion mapping

In the notation introduced in Part I, Sec. III.A, the collective diquark annihilation operator $A_r$ is given by

$$A_r =: \sqrt{2\Omega} \sum_{ab} \sum_{\alpha\beta} C^{r*}_{a\alpha b\beta}\, a_{b\beta} a_{a\alpha} \,, \tag{3.1}$$

where the coupling coefficients

$$C^r_{a\alpha b\beta} := 1/\sqrt{2\Omega}\ \varepsilon_{abr}\, g_{\alpha\beta} \tag{3.2}$$

are antisymmetric with respect to an interchange of the double indices $(a\alpha)$ and $(b\beta)$ and obey the orthogonality relation (I.3.3). The colour index r appearing here is a collective difermion index in the sense of Part I, Sec. IV.A. With the aid of Eq. (I.3.1), $A_r$ takes on the familiar form

$$A_r = 2\sqrt{2\Omega}\ (aa)_r \ . \tag{3.3}$$

In a completely analogous way, the operator $N_\rho$ annihilating a nucleon, i. e., a collective triquark, is written as

$$N_\rho =: 2\sqrt{6\Omega} \sum_{rc} \sum_{\gamma} D^{\rho *}_{rc\gamma}\, a_{c\gamma}(aa)_r \tag{3.4}$$

$$= \sqrt{6\Omega} \sum_{\substack{ab\\ rc}} \sum_{\alpha\beta\gamma} C^{r*}_{a\alpha b\beta}\, D^{\rho *}_{rc\gamma}\, a_{c\gamma} a_{b\beta} a_{a\alpha} \tag{3.5}$$

with the coefficients

$$D^{\rho}_{rc\gamma} := 1/\sqrt{3}\ \delta_{cr}\delta_{\gamma\rho}\ . \tag{3.6}$$

In this definition, $\rho$ is the collective trifermion index introduced in Sec. IV.A of Part I, and the indices $(c\gamma)$ and $r$ stand for the contributing single-fermion and collective difermion states, respectively. The coefficients $D^{\rho}_{rc\gamma}$ fulfill the orthogonality relation (I.3.8). (There, due to the property (I.4.1), terms with noncollective difermion indices do not appear in the sum.) Using the notation of Eqs. (I.4.2) and (I.4.4), one obtains from (3.4):

$$N_\rho = 2\sqrt{6\Omega}\ (aaa)_\rho = 2\sqrt{6\Omega}\ T_\rho\ . \tag{3.7}$$

The corresponding extended image $(N_\rho)_{\mathcal{T}}$, constructed by means of Eqs. (I.3.16), (I.4.3), and (I.3.10), is given by

$$(N_\rho)_{\mathcal{T}} = 2\sqrt{6\Omega}\ (bc)_\rho \tag{3.8}$$

with the collective boson-fermion pair

$$(bc)_\rho = \sum_{rc} \sum_{\gamma} D^{\rho *}_{rc\gamma}\, b_r c_{c\gamma} \tag{3.9}$$

$$= 1/\sqrt{3} \sum_{r} b_r c_{r\rho} \tag{3.10}$$

and the collective boson operators

$$b_r = 1/2 \sum_{ab} \sum_{\alpha\beta} C^{r*}_{a\alpha b\beta}\, b_{a\alpha b\beta} \tag{3.11}$$

$$= \frac{1}{2\sqrt{2\Omega}} \sum_{ab} \sum_{\alpha\beta} \varepsilon_{abr}\, g_{\alpha\beta}\, b_{a\alpha b\beta} \ . \tag{3.12}$$

According to Part I, Sec. IV.B, the collective ideal space $\mathcal{I}_C$ is then spanned by the states

$$\left.\begin{array}{l} (N_{\rho_n})^{+}_{\mathcal{I}} \ldots (N_{\rho_1})^{+}_{\mathcal{I}}\ |0) \propto (b^+c^+)_{\rho_n} \ldots (b^+c^+)_{\rho_1}|0) \\ \text{with} \quad n = 0,1,\ldots,2\Omega \ , \\ \qquad\quad \rho_1,\ldots,\rho_n = 1,\ldots,2\Omega \ , \\ \qquad\quad \rho_i \neq \rho_j \quad \text{for} \quad i,\ j = 1,\ldots,n \quad \text{and} \quad i \neq j \ . \end{array}\right\} \tag{3.13}$$

### B. Baryon mapping

The nucleon annihilation operator is now expressed in the form

$$N_\rho =: \frac{\sqrt{6(\Omega+1)}}{3} \sum_{abc} \sum_{\alpha\beta\gamma} C^{\rho*}_{a\alpha b\beta c\gamma}\, a_{c\gamma} a_{b\beta} a_{a\alpha} \tag{3.14}$$

with the collective triquark coupling coefficients

$$C^{\rho}_{a\alpha b\beta c\gamma} := \frac{1}{\sqrt{6(\Omega+1)}}\, \varepsilon_{abc}\, (g_{\alpha\beta}\, \delta_{\gamma\rho} + g_{\gamma\alpha}\, \delta_{\beta\rho} + g_{\beta\gamma}\, \delta_{\alpha\rho}) \tag{3.15}$$

introduced in Sec. III.B of Part I. Here, $\rho$ is a collective trifermion index as in Part I, Sec. IV.A. The coefficients $C^{\rho}_{a\alpha b\beta c\gamma}$ fulfill the requirement of total antisymmetry with respect to permutations of the three double indices $(a\alpha)$, $(b\beta)$, and $(c\gamma)$ and are normalized so as to satisfy the orthogonality relation

(I.3.24). By inserting Eq. (I.3.22) into (3.14) and using (I.4.4), one derives the representations

$$N_\rho = 2\sqrt{6(\Omega+1)}\,(aaa)_\rho = 2\sqrt{6(\Omega+1)}\,T_\rho \tag{3.16}$$

of $N_\rho$.

The extended image $(N_\rho)_{\mathcal{J}}$ of $N_\rho$, constructed by means of Eq. (I.3.31), is then found to be

$$(N_\rho)_{\mathcal{J}} = 2\sqrt{6(\Omega+1)}\,b_\rho\,, \tag{3.17}$$

where the collective ideal baryon operators are given by (I.3.26), i. e.,

$$b_\rho = 1/6 \sum_{abc} \sum_{\alpha\beta\gamma} C^{\rho *}_{a\alpha b\beta c\gamma}\, b_{a\alpha b\beta c\gamma}\,. \tag{3.18}$$

When the total antisymmetry of $\varepsilon_{abc}$ in Eq. (3.15) is made use of, this expression can be written in the simple form

$$b_\rho = \frac{1}{2\sqrt{6(\Omega+1)}} \sum_{abc} \sum_{\alpha\beta} \varepsilon_{abc}\, g_{\alpha\beta}\, b_{a\alpha b\beta c\rho}\,. \tag{3.19}$$

The collective ideal space $\mathcal{J}_C$ is now spanned by the states

$$\left.\begin{array}{l} (N_{\rho_n})^+_{\mathcal{J}} \cdots (N_{\rho_1})^+_{\mathcal{J}}\,|0) \propto b^+_{\rho_n}\cdots b^+_{\rho_1}|0) \\ \text{with}\quad n = 0,1,\ldots,2\Omega\,, \\ \qquad\quad \rho_1,\ldots,\rho_n = 1,\ldots,2\Omega\,, \\ \qquad\quad \rho_i \neq \rho_j \quad \text{for}\quad i,\, j = 1,\ldots,n \quad \text{and} \quad i \neq j\,. \end{array}\right\} \tag{3.20}$$

# IV. The model space

The aim in this section is to find a collective extended image $(N_\rho^+)'_{\mathcal{J}}$ of the nucleon creation operator $N_\rho^+$ and then to construct the states of the model space $\mathcal{J}_{CPC}$ according to the scheme of Part I, Sec. V.B. As a starting point, the *exact* image $(N_\rho^+)_{\mathcal{P}}$ of $N_\rho^+$ is calculated in Sec. IV.A. In the following step, a collective extended image $(N_\rho^+)'_{\mathcal{J}}$ is derived with the aid of the fundamental relation (I.5.8). The algorithm used for this purpose is outlined in Sec. IV.B, and the results obtained with the boson-fermion and the baryon mapping are presented in Secs. IV.C and IV.D, respectively.

### A. Exact image of the nucleon creation operator

In the case of the boson-fermion mapping, the image $(N_\rho^+)_{\mathcal{P}}$ of $N_\rho^+$ is derived by means of Eq. (3.5), where the coefficients (3.6) are inserted. The result reads:

$$(N_\rho^+)_{\mathcal{P}}$$

$$= \sqrt{2\Omega} \sum_{abr} \sum_{\alpha\beta} C^r_{a\alpha b\beta} \, (a^+_{a\alpha} a^+_{b\beta} a^+_{r\rho})_{\mathcal{P}} \tag{4.1}$$

$$\begin{aligned}
= \sqrt{2\Omega} \sum_{abr} \sum_{\alpha\beta} C^r_{a\alpha b\beta} \\
\Big( \; & b^+_{a\alpha b\beta} c^+_{r\rho} + b^+_{b\beta r\rho} c^+_{a\alpha} + b^+_{r\rho a\alpha} c^+_{b\beta} \\
& + \sum_{k} \sum_{\varkappa} \big( \; b^+_{a\alpha k\varkappa} c^+_{b\beta} c^+_{r\rho} \\
& \qquad + b^+_{b\beta k\varkappa} c^+_{r\rho} c^+_{a\alpha} \\
& \qquad + b^+_{r\rho k\varkappa} c^+_{a\alpha} c^+_{b\beta} \big) \; c_{k\varkappa} \\
& + \sum_{kl} \sum_{\varkappa\lambda} \big( \; b^+_{a\alpha l\lambda} b^+_{b\beta k\varkappa} c^+_{r\rho} \\
& \qquad + b^+_{b\beta l\lambda} b^+_{r\rho k\varkappa} c^+_{a\alpha} \\
& \qquad + b^+_{r\rho l\lambda} b^+_{a\alpha k\varkappa} c^+_{b\beta} \big) \; b_{k\varkappa l\lambda} \\
& + \Big( \; \sum_{kl} \sum_{\varkappa\lambda} \big( \; b^+_{a\alpha b\beta} b^+_{r\rho k\varkappa} \\
& \qquad + b^+_{b\beta r\rho} b^+_{a\alpha k\varkappa} \\
& \qquad + b^+_{r\rho a\alpha} b^+_{b\beta k\varkappa} \big) \; c^+_{l\lambda} b_{k\varkappa l\lambda} \\
& \quad + 1/2 \sum_{kl} \sum_{\varkappa\lambda} b^+_{k\varkappa l\lambda} c^+_{a\alpha} c^+_{b\beta} c^+_{r\rho} c_{l\lambda} c_{k\varkappa} \\
& \quad + 1/3 \sum_{\substack{kl \\ ij}} \sum_{\substack{\varkappa\lambda \\ \mu\nu}} b^+_{a\alpha k\varkappa} b^+_{b\beta l\lambda} b^+_{r\rho i\mu} c^+_{j\nu} \\
& \qquad \big( \; b_{k\varkappa l\lambda} b_{j\nu i\mu} \\
& \qquad + b_{l\lambda i\mu} b_{j\nu k\varkappa} \\
& \qquad + b_{i\mu k\varkappa} b_{j\nu l\lambda} \big) \Big) \frac{1}{\hat{N}+1} \Big) \hat{P} \, .
\end{aligned} \tag{4.2}$$

Here, Eqs. (I.3.14) - (I.3.15) have been used for the construction of the image operator $(a^+_{a\alpha}a^+_{b\beta}a^+_{r\rho})_{\mathcal{P}}$. The terms of the last sum have been rewritten in such a way that the resulting formula is easier to handle in the process of truncation. The operator $\hat{N}$ appearing on the right side of Eq. (4.2) is the particle number operator of the bosons or ideal fermions, as explained in Sec. III.A of Part I. As usual, the symbol $\hat{P}$ represents the orthogonal projection operator onto the physical subspace $\mathcal{P}$.

In the case of the baryon mapping, the image $(N^+_\rho)_{\mathcal{P}}$ of $N^+_\rho$ is found to be

$$\begin{aligned}
&(N^+_\rho)_{\mathcal{P}} \\
&= \frac{\sqrt{6(\Omega+1)}}{3} \sum_{abc} \sum_{\alpha\beta\gamma} C^{\rho}_{a\alpha b\beta c\gamma} \, (a^+_{a\alpha}a^+_{b\beta}a^+_{c\gamma})_{\mathcal{P}} \qquad (4.3) \\
&= \frac{\sqrt{6(\Omega+1)}}{3} \sum_{abc} \sum_{\alpha\beta\gamma} C^{\rho}_{a\alpha b\beta c\gamma} \\
&\qquad \Big( \, b^+_{a\alpha b\beta c\gamma} \\
&\qquad + 1/2 \sum_{kli} \sum_{\varkappa\lambda\mu} \big( \, b^+_{k\varkappa l\lambda a\alpha} b^+_{b\beta c\gamma i\mu} \\
&\qquad\qquad + b^+_{k\varkappa l\lambda b\beta} b^+_{c\gamma a\alpha i\mu} \\
&\qquad\qquad + b^+_{k\varkappa l\lambda c\gamma} b^+_{a\alpha b\beta i\mu} \big) \\
&\qquad\qquad b_{k\varkappa l\lambda i\mu} \\
&\qquad + 1/8 \sum_{\substack{kix \\ ljy}} \sum_{\substack{\varkappa\mu\xi \\ \lambda\nu\eta}} b^+_{k\varkappa l\lambda a\alpha} b^+_{i\mu j\nu b\beta} b^+_{x\xi y\eta c\gamma} \\
&\qquad\qquad b_{k\varkappa i\mu x\xi} b_{l\lambda j\nu y\eta} \, \Big) \, \hat{P} \, . \qquad (4.4)
\end{aligned}$$

This expression has been derived with the aid of Eqs. (3.14) and (I.3.30).

### B. Construction of a collective extended image

For the operator $N_p^+$, a collective extended image $(N_p^+)'_{\mathcal{I}}$ as defined in Sec. V of Part I can be constructed according to the algorithm described below. The method is applicable to both the boson-fermion and the baryon mapping. The relation

$$(N_p^+)_{\mathcal{P}} = (N_p^+)_{\mathcal{I}}\ \hat{P} \tag{4.5}$$

obtained from Eq. (I.2.4) serves as the starting point. The variant of the extended image $(N_p^+)_{\mathcal{I}}$ is arbitrary but has to be normal-ordered. Here, the expression given in Sec. IV.A is inserted for $(N_p^+)_{\mathcal{I}}\ \hat{P}$. The equation is then multiplied on the left by the orthogonal projection operator $\hat{C}$ onto $\mathcal{I}_C$ and on the right by the orthogonal projector $\hat{P}_C$ onto the physical collective subspace $\mathcal{P}_C = \hat{P}\ \mathcal{I}_C$ (see Part I, Sec. IV.B), yielding

$$\hat{C}\ (N_p^+)_{\mathcal{P}}\ \hat{P}_C = \hat{C}\ (N_p^+)_{\mathcal{I}}\ \hat{P}_C\ . \tag{4.6}$$

(For simplicity, the property (I.4.14) of $\hat{P}_C$ has been used to eliminate the factor $\hat{P}$ originating from Eq. (4.5).) The ultimate aim is to bring the right side of (4.6) into the form presented in Eq. (I.5.8),

$$\hat{C}\ (N_p^+)_{\mathcal{P}}\ \hat{P}_C = (N_p^+)'_{\mathcal{I}}\ \hat{C}\ \hat{P}_C\ , \tag{4.7}$$

where $(N_p^+)'_{\mathcal{I}}$ is an operator containing no noncollective operators of bosons or of ideal baryons. It is easier first to fulfill the equation

$$\hat{C}\ (N_p^+)_{\mathcal{P}}\ \hat{P}_C = \hat{C}\ (N_p^+)'_{\mathcal{I}}\ \hat{P}_C\ . \tag{4.8}$$

For this purpose, the following two tricks will be used:

a) From the *creation* operators of the normal-ordered extended image $(N_\rho^+)_\mathcal{J}$ in Eq. (4.6), all components ineffective in the presence of the projection operator $\hat{C}$ are to be removed.

b) As shown in Refs. 2 and 3, any normal-ordered extended image multiplied by $\hat{P}$ on the right is totally antisymmetric in the fermion indices of its *annihilation* operators. Due to the property (I.4.14), $\hat{P}_C$ has the same effect. Consequently, one can further modify the variant of $(N_\rho^+)_\mathcal{J}$ in (4.6) by rearranging the indices of the annihilation operators.

In detail, one proceeds as follows:

1. On the right side of Eq. (4.6), the *creation* operators of the bosons or ideal baryons in $(N_\rho^+)_\mathcal{J}$ are expressed in terms of operators with coupled fermion indices as described in Part I, Sec. III. The noncollective components of these creation operators yield no contributions in the presence of $\hat{C}$ and are therefore dropped according to trick a). In this way, the extended image $(N_\rho^+)_\mathcal{J}$ in Eq. (4.6) is reduced to the terms where all boson or ideal baryon creation operators are collective. The fermion indices of the *annihilation* operators remain uncoupled.

2. In the second step, trick b) is applied in order to further simplify the structure of $(N_\rho^+)_\mathcal{J}$ and to bring the annihilation operators as far as possible into a purely collective form.

3. In the example considered here, the case of the boson-fermion mapping requires an additional treatment because the ideal fermions and col-

lective bosons in the states of $\mathcal{J}_C$ are coupled to higher collective units, the boson-fermion pairs. As a consequence, the procedure of step 1 does not suffice to exclude all terms annihilated by $\hat{C}$ from the expression $(N_\rho^+)_{\mathcal{J}}\ \hat{P}_C$. It is necessary that the operator $(N_\rho^+)_{\mathcal{J}}\ \hat{P}_C$ simplified in steps 1 and 2 is explicitly applied to a collective bra state with the general structure

$$(0|\ (N_{\sigma_1})_{\mathcal{J}}\ \dots\ (N_{\sigma_n})_{\mathcal{J}}\ .$$

The resulting state is physical and therefore completely antisymmetric in the original fermion indices of its annihilation operators. Making use of this property, one can show that the remaining noncollective components of $(N_\rho^+)_{\mathcal{J}}$ contained in $\hat{C}\ (N_\rho^+)_{\mathcal{J}}\ \hat{P}_C$ give zero when acting on collective bra states and may be dropped. The rest is purely collective.

For the purpose of an additional simplification, the procedure just described can be applied also to the *collective* parts of the operator $(N_\rho^+)_{\mathcal{J}}$. Here, all terms for which the result is zero are omitted from $(N_\rho^+)_{\mathcal{J}}$.

4. For both mapping techniques, the extended image $(N_\rho^+)_{\mathcal{J}}$ in the expression $\hat{C}\ (N_\rho^+)_{\mathcal{J}}\ \hat{P}_C$ has been reduced in the preceding steps to an operator $(N_\rho^+)'_{\mathcal{J}}$ which contains no more noncollective operators of bosons or of ideal baryons and satisfies Eq. (4.8). Then, Eq. (4.7) is fulfilled as well if the operator $(N_\rho^+)'_{\mathcal{J}}$ commutes with $\hat{C}$ or, which is equivalent but easier to verify, obeys the two conditions

$$\left.\begin{aligned} (N_\rho^+)'_{\mathcal{J}}\ |c) &= \hat{C}\ (N_\rho^+)'_{\mathcal{J}}\ |c) \qquad \forall\ \ |c) \in \mathcal{J}_C\ , \\ (c|\ (N_\rho^+)'_{\mathcal{J}} &= (c|\ (N_\rho^+)'_{\mathcal{J}}\ \hat{C} \qquad \forall\ \ |c) \in \mathcal{J}_C\ . \end{aligned}\right\} \quad (4.9)$$

In the case of the baryon mapping, any operator containing only collective ideal baryon operators automatically satisfies (4.9). For the boson-fermion mapping, however, the situation is in general not that simple because the coupling between bosons and ideal fermions renders the structure of $\mathcal{J}_C$ comparatively complicated. Therefore, the validity of Eqs. (4.9) has to be checked separately. It is sufficient to do this for a state $|c)$ of the general form (3.13). In this way, the operator $(N^+_\rho)'_{\mathcal{J}}$ can be shown to commute with $\hat{C}$ and, consequently, to possess the desired property (4.7). From this it follows that $(N^+_\rho)'_{\mathcal{J}}$ is a collective extended image.

### C. Results for the boson-fermion mapping

Let

$$\hat{N}_{\mathcal{C}} := \sum_{k} \sum_{\varkappa} c^+_{k\varkappa} c_{k\varkappa} \tag{4.10}$$

be the particle number operator of the ideal fermions and

$$\hat{N}'_{\mathcal{B}} := \sum_{k} b^+_k b_k \tag{4.11}$$

that of the *collective* bosons, where the prime indicates the truncation. With these definitions, the collective extended boson-fermion image $(N^+_\rho)'_{\mathcal{J}}$ constructed according to Sec. IV.B reads:

$$\begin{aligned} (N^+_\rho)'_{\mathcal{J}} &= \frac{1}{2\Omega} \Big( \; (N_\rho)^+_{\mathcal{J}} \, (2\Omega + 2 - \hat{N}'_{\mathcal{B}} - \hat{N}_{\mathcal{C}}) \\ &\qquad - \sum_{\varkappa} (N_\varkappa)^+_{\mathcal{J}} \sum_k c^+_{k\rho} c_{k\varkappa} \Big) \qquad (4.12) \\ &= \sqrt{6/\Omega} \, \Big( \; (b^+c^+)_\rho \, (2\Omega + 2 - \hat{N}'_{\mathcal{B}} - \hat{N}_{\mathcal{C}}) \\ &\qquad - \sum_{\varkappa} (b^+c^+)_\varkappa \sum_k c^+_{k\rho} c_{k\varkappa} \Big) \; . \qquad (4.13) \end{aligned}$$

(In the derivation of the last equation, the relation (3.8) has been used.)

By applying the operator (4.13) to a ket state of the form (3.13), one obtains the recursive relation

$$\begin{aligned} &(N^+_{\rho_{n+1}})'_{\mathcal{J}} \; (b^+c^+)_{\rho_n} \ldots (b^+c^+)_{\rho_1} |0) \\ &= \sqrt{6/\Omega} \; (2\Omega+2-n) \; (b^+c^+)_{\rho_{n+1}} (b^+c^+)_{\rho_n} \ldots (b^+c^+)_{\rho_1} |0) \; . \qquad (4.14) \end{aligned}$$

As demonstrated in Sec. V.B of Part I, the model space $\mathcal{J}_{CPC}$ is spanned by the states

$$\left. \begin{array}{l} (N^+_{\rho_n})'_{\mathcal{J}} \; \ldots \; (N^+_{\rho_1})'_{\mathcal{J}} \; |0) \\ \text{with} \quad n = 0,1,\ldots,2\Omega \; , \\ \qquad\quad \rho_1,\ldots,\rho_n = 1,\ldots,2\Omega \; , \\ \qquad\quad \rho_i \neq \rho_j \quad \text{for} \quad i,\, j = 1,\ldots,n \quad \text{and} \quad i \neq j \; , \end{array} \right\} \qquad (4.15)$$

which are the collective images of the multinucleon states (2.10) for $n = 0,1,\ldots,2\Omega$. With the aid of the formula (4.14), the states (4.15) can be reduced to a very simple structure. After repeated use of Eq. (4.14), one arrives at the relation

$$(N^+_{\rho_n})'_{\mathcal{J}} \cdots (N^+_{\rho_1})'_{\mathcal{J}} \, |0)$$

$$= (6/\Omega)^{n/2} \frac{(2\Omega+2)!}{(2\Omega+2-n)!} (b^+c^+)_{\rho_n} \ldots (b^+c^+)_{\rho_1}|0) \; . \qquad (4.16)$$

A comparison of (4.15) and (4.16) with (3.13) shows that in the special case of the model treated here, the model space $\mathcal{J}_{CPC}$ is *identical* with the collective ideal space $\mathcal{J}_C$,

$$\mathcal{J}_{CPC} = \mathcal{J}_C \; . \qquad (4.17)$$

This property leads to a considerable simplification of practical calculations because a collective analogue of the *un*physical subspace $\mathcal{P}^\perp$, i. e., a nontrivial orthogonal complement of $\mathcal{J}_{CPC}$ in $\mathcal{J}_C$, does not exist.

### D. Results for the baryon mapping

With the particle number operator

$$\hat{N}' := \sum_{\varkappa} b^+_{\varkappa} b_{\varkappa} \qquad (4.18)$$

of the *collective* ideal baryons, the collective extended baryon image $(N^+_\rho)'_{\mathcal{J}}$ constructed in Sec. IV.B can be written as

$$(N^+_\rho)'_{\mathcal{J}} = (N_\rho)^+_{\mathcal{J}} \left(1 + \frac{1}{6(\Omega+1)} \hat{N}' \, (2\Omega - 1 - \hat{N}')\right) \qquad (4.19)$$

$$= 2\sqrt{6(\Omega+1)} \; b^+_\rho \left(1 + \frac{1}{6(\Omega+1)} \hat{N}' \, (2\Omega - 1 - \hat{N}')\right) . \qquad (4.20)$$

Here, the relation (3.17) between $(N_p)^+_{\mathcal{J}}$ and $b^+_p$ has been used.

For collective ket states with the structure (3.20), one obtains the recursion formula

$$(N^+_{p_{n+1}})'_{\mathcal{J}}\; b^+_{p_n}\dots b^+_{p_1}|0)$$

$$= \frac{2}{\sqrt{6(\Omega+1)}}\Big(6(\Omega+1) + n(2\Omega-1-n)\Big)\; b^+_{p_{n+1}}b^+_{p_n}\dots b^+_{p_1}|0)\ . \tag{4.21}$$

As a consequence, the collective multinucleon image states (4.15) spanning the model space $\mathcal{J}_{CPC}$ are now found to be

$$(N^+_{p_n})'_{\mathcal{J}}\ \dots\ (N^+_{p_1})'_{\mathcal{J}}\ |0)$$

$$\begin{aligned} = 2^n \left(\frac{1}{6(\Omega+1)}\right)^{n/2} \quad & 6(\Omega+1) \\ & \times \Big(6(\Omega+1) + 1(2\Omega-2)\Big) \\ & \times \Big(6(\Omega+1) + 2(2\Omega-3)\Big) \\ & \times \dots \\ & \times \Big(6(\Omega+1) + (n-1)(2\Omega-n)\Big)\; b^+_{p_n}\dots b^+_{p_1}|0)\ . \end{aligned} \tag{4.22}$$

Again, the model shows the special feature that the subspace $\mathcal{J}_{CPC}$ is *identical* with the whole collective ideal space $\mathcal{J}_C$,

$$\mathcal{J}_{CPC} = \mathcal{J}_C\ . \tag{4.23}$$

## V. Mapping of the pairing force operator

According to the procedure described in Sec. IV for the operator $N_p^+$, a collective extended image $\hat{H}'_{\mathcal{J}}$ of the pairing force operator $\hat{H}$ defined in Eq. (2.9) is constructed. The calculation starts with the exact image $\hat{H}_{\mathcal{P}}$, which will be presented in Sec. V.A. The same algorithm as in Sec. IV.B is applied. The results are given in Sec. V.B.

### A. Exact image

In the case of the boson-fermion mapping, Eq. (4.32a) of Ref. 2 leads to the result

$$\begin{aligned}\hat{H}_{\mathcal{P}} = G/4 \sum_{\substack{abr\\cd}} \sum_{\substack{\alpha\beta\\\gamma\delta}} \varepsilon_{abr}\varepsilon_{cdr}\, g_{\alpha\beta} g_{\gamma\delta} \Big( & c^+_{a\alpha} c^+_{b\beta} c_{c\gamma} c_{d\delta} + b^+_{a\alpha b\beta} b_{d\delta c\gamma} \\ & - \sum_{kl} \sum_{\varkappa\lambda} b^+_{a\alpha k\varkappa} b^+_{b\beta l\lambda} b_{k\varkappa l\lambda} b_{d\delta c\gamma} \\ & - \sum_{k} \sum_{\varkappa} ( b^+_{a\alpha k\varkappa} c^+_{b\beta} \\ & \qquad - b^+_{b\beta k\varkappa} c^+_{a\alpha} )\, b_{d\delta c\gamma} c_{k\varkappa} \Big) \hat{P}\ . \qquad (5.1)\end{aligned}$$

The analogous expression for the baryon mapping is derived from Eq. (5.26a) of Ref. 2 and reads:

$$\hat{H}_{\mathcal{P}} = G/4 \sum_{\substack{abr \\ cd}} \sum_{\substack{\alpha\beta \\ \gamma\delta}} \varepsilon_{abr}\varepsilon_{cdr}\; g_{\alpha\beta} g_{\gamma\delta} \Big( \sum_{k} \sum_{\varkappa} b^{+}_{a\alpha b\beta k\varkappa} b_{d\delta c\gamma k\varkappa} + 1/4 \sum_{\substack{kl \\ ij}} \sum_{\substack{\varkappa\lambda \\ \mu\nu}} b^{+}_{k\varkappa l\lambda a\alpha} b^{+}_{i\mu j\nu b\beta}\, b_{k\varkappa l\lambda j\nu} b_{d\delta c\gamma i\mu} \Big) \hat{P} \,. \tag{5.2}$$

**B. Collective extended image**

For the boson-fermion mapping, the algorithm of Sec. IV.B leads to the collective extended image

$$\hat{H}'_{\mathcal{J}} = -\, G \left( \hat{N}'_{\mathcal{B}} \,(2\Omega + 1 - \hat{N}'_{\mathcal{B}} - \hat{N}_{\mathcal{C}}) + \frac{1}{8\Omega} \sum_{\varkappa} (N_{\varkappa})^{+}_{\mathcal{J}}\, (N_{\varkappa})_{\mathcal{J}} \right) \tag{5.3}$$

$$= -\, G \left( \hat{N}'_{\mathcal{B}} \,(2\Omega + 1 - \hat{N}'_{\mathcal{B}} - \hat{N}_{\mathcal{C}}) + 3 \sum_{\varkappa} (b^{+}c^{+})_{\varkappa}(bc)_{\varkappa} \right) . \tag{5.4}$$

This expression is identical with the operator given by Pittel et al. in Eqs. (6) - (7) of Ref. 6 although the authors use a different mapping technique.

For the baryon mapping, the collective extended image derived according to the scheme of Sec. IV.B is

$$\hat{H}'_{\mathcal{J}} = -\, G\, \hat{N}' \,(2\Omega + 3 - \hat{N}') \,. \tag{5.5}$$

In spite of the different definition of the single-particle operators (see also Sec. VI), this formula has the same structure as the result obtained by Pittel et al. in terms of triplet fermions (Eq. (11) of Ref. 6).

## VI. Comparison with the method of Pittel et al.

In this section, the boson-fermion and the baryon mapping are contrasted with the method of Pittel, Engel, Dukelsky, and Ring.[6] In spite of the obvious similarities, there are some essential differences, which will be analyzed in the following. As long as the discussion does not refer to a special physical model, the general notation of Ref. 2 and Part I will be used, where, contrary to the case of the quark shell model, the *whole* set of quantum numbers characterizing the state of a quark is labeled by a single greek index.

In the first step of the iterative mapping technique employed by Pittel et al., the ideal space consists of states containing bosons and ideal fermions. The Dyson boson mapping for systems with an odd number of fermions[8] is applied to the bilinear operators of the form

$$a^+_\alpha a^+_\beta \ , \ a_\beta a_\alpha \ , \ a^+_\alpha a_\beta \tag{6.1}$$

and leads to a representation in terms of boson and ideal fermion operators.

In the next step, a second nonunitary mapping, acting on the bilinear operators

$$b^+_{\alpha\beta} c^+_\gamma \ , \ b_{\alpha\beta} c_\gamma \ , \ b^+_{\alpha\beta} b_{\gamma\delta} \ , \ c^+_\alpha c_\beta \ , \tag{6.2}$$

introduces a new kind of fermions with three-quark quantum numbers. These so-called *triplet fermions* are described by the creation and annihilation operators $c^+_{\alpha\beta,\gamma}$ and $c_{\alpha\beta,\gamma}$, which, unlike the ideal baryon operators, are not completely antisymmetric in their indices but possess the symmetry properties of the boson-fermion systems created in the previous step. This is expressed by the equations

$$c^+_{\alpha\beta,\gamma} = - c^+_{\beta\alpha,\gamma} \, , \tag{6.3}$$

$$\{c_{\alpha\beta,\gamma}, c^+_{\varkappa\lambda,\mu}\} = (\delta_{\alpha\varkappa}\delta_{\beta\lambda} - \delta_{\beta\varkappa}\delta_{\alpha\lambda})\, \delta_{\gamma\mu} \tag{6.4}$$

(cf. Sec. II of Part I). The ideal space is now spanned by the many-particle states of the triplet fermions.

Whereas the boson-fermion and the baryon mapping are derived according to the Marumori method,[11-13] which is based on the transformation of the multifermion space $\mathcal{F}$, the approach used by Pittel et al. starts immediately with a mapping of operators. Following the method of Beliaev and Zelevinsky,[13,14] the construction of the operator images is determined by the criterion that the commutation relations of bilinear operators be preserved at each stage of the two-step procedure. Pittel et al. do not attempt to deduce an exact image of the original multiquark Hilbert space. In the first step of the formalism, where the conventional Dyson transformation is applied, such a physical subspace would in general not be invariant under the resulting image operators. An example is given by the Dyson image[8,6]

$$(a^+_\alpha a^+_\beta a_\gamma a_\delta)_1 := (a^+_\alpha a^+_\beta)_1 \, (a_\gamma a_\delta)_1 \tag{6.5}$$

$$\begin{aligned} = \quad & b^+_{\alpha\beta} b_{\delta\gamma} - \sum_{\varkappa\lambda} b^+_{\alpha\varkappa} b^+_{\beta\lambda} b_{\varkappa\lambda} b_{\delta\gamma} \\ & - \sum_{\varkappa} (b^+_{\alpha\varkappa} c^+_\beta - b^+_{\beta\varkappa} c^+_\alpha)\, b_{\delta\gamma} c_\varkappa \end{aligned} \tag{6.6}$$

of the operator $a^+_\alpha a^+_\beta a_\gamma a_\delta$, where the subscript 1 indicates the first step of the mapping. An extended image possessing a similar structure can be constructed with the boson-fermion mapping (see Eq. (4.32a) of Ref. 2):

$$(a^+_\alpha a^+_\beta a_\gamma a_\delta)_{\mathcal{J}} = \quad c^+_\alpha c^+_\beta c_\gamma c_\delta + b^+_{\alpha\beta} b_{\delta\gamma} - \sum_{\varkappa\lambda} b^+_{\alpha\varkappa} b^+_{\beta\lambda} b_{\varkappa\lambda} b_{\delta\gamma}$$

$$- \sum_{\varkappa} (b^+_{\alpha\varkappa} c^+_\beta - b^+_{\beta\varkappa} c^+_\alpha)\, b_{\delta\gamma} c_\varkappa \ . \qquad (6.7)$$

According to Eq. (I.2.6), this operator leaves the space of physical ket states invariant. The expression (6.6), on the contrary, does not have this property because the term $c^+_\alpha c^+_\beta c_\gamma c_\delta$ is missing. When the operator $(a^+_\alpha a^+_\beta a_\gamma a_\delta)_1$ given by Eq. (6.6) is applied to a physical state of the form

$$\hat{P}\ b^+_{\xi\eta} c^+_\zeta b^+_{\rho\sigma} c^+_\tau |0) \ ,$$

the result consists of terms where the indices $\alpha$ and $\beta$ belong either to the same boson or to two different bosons or to a boson and an ideal fermion. A component containing $c^+_\alpha c^+_\beta$ cannot arise. As a consequence, the resulting state is not totally antisymmetric in its six indices and therefore not physical. The invariance of the physical subspace is destroyed because the conventional Dyson mapping designed for the representation of pair structures in the presence of a *single* unpaired fermion[8] is not suited for the case of *many* ideal fermions.

As already mentioned, Pittel et al. have applied their method to the quark shell model. In the following, the central results obtained for a nucleon system in the single-orbit version of the model are summarized and compared with those of the boson-fermion and the baryon mapping. From now on, the special notation introduced in Secs. II and III for the states of single quarks and of collective di- and triquarks will be used again.

In the first step, the pairing force operator $\hat{H}$ in Eq. (2.9) is transformed into the operator

$$\hat{H}_1 := G/4 \sum_{\substack{abr\\cd}} \sum_{\substack{\alpha\beta\\\gamma\delta}} \varepsilon_{abr}\varepsilon_{cdr}\, g_{\alpha\beta}g_{\gamma\delta}\, (a^+_{a\alpha}a^+_{b\beta})_1\, (a_{c\gamma}a_{d\delta})_1 \,, \tag{6.8}$$

where the expression $(a^+_{a\alpha}a^+_{b\beta})_1\ (a_{c\gamma}a_{d\delta})_1$ has the structure shown in Eq. (6.6). All terms containing noncollective boson components are then removed from $\hat{H}_1$. The resulting truncated operator $\hat{H}'_1$ (with the prime indicating the truncation) is given by Eqs. (6) - (7) of Ref. 6 and can be written in the form

$$\hat{H}'_1 := -\, G\left(\hat{N}'_{\mathcal{B}}\,(2\Omega + 1 - \hat{N}'_{\mathcal{B}} - \hat{N}_{\mathcal{C}}) + 3\sum_{\varkappa} (b^+c^+)_\varkappa (bc)_\varkappa\right) , \tag{6.9}$$

which is identical with the collective extended boson-fermion image (5.4).

In the second step, the operators $\hat{N}'_{\mathcal{B}}$, $\hat{N}_{\mathcal{C}}$, $(b^+c^+)_\varkappa$, and $(bc)_\varkappa$ in $\hat{H}'_1$ are expressed in terms of triplet fermions. Collective triplet fermion operators can be defined by the equation

$$c^+_\rho := \frac{1}{2\sqrt{6\Omega}} \sum_{abr} \sum_{\alpha\beta} \varepsilon_{abr}\, g_{\alpha\beta}\, c^+_{a\alpha b\beta, r\rho} \,. \tag{6.10}$$

The creation and annihilation operators introduced in this way obey ordinary fermionic anticommutation rules. After the truncation to purely collective triplet fermion operators, the image of $\hat{H}$ is given by

$$\hat{H}'_2 := -\, G \sum_{\varkappa} c^+_\varkappa c_\varkappa \left(2\Omega + 3 - \sum_{\lambda} c^+_\lambda c_\lambda\right) \tag{6.11}$$

(see Eq. (11) of Ref. 6), where the subscript 2 stands for the second step of the mapping. The operator

$$\sum_{\varkappa} c^+_\varkappa c_\varkappa \tag{6.12}$$

is the particle number operator of the collective triplet fermions. As mentioned before, there is a striking similarity between $\hat{H}'_2$ and the collective extended baryon image (5.5).

It has been demonstrated above that the physical subspace of the ideal boson-fermion space is not invariant under the exact image $\hat{H}_1$ constructed in the first step of the formalism of Ref. 6. This deficiency, however, is repaired when the truncation is performed. For the case of the boson-fermion transformation, the model space $\mathcal{J}_{CPC}$ has been shown to be identical with the whole collective ideal space $\mathcal{J}_C$. Since the collective extended boson-fermion image $\hat{H}'_{\mathcal{J}}$ given by Eq. (5.4) leaves the model space invariant, the same must be true also for the truncated operator $\hat{H}'_1$, which agrees with $\hat{H}'_{\mathcal{J}}$. In this way, the invariance of the image space is restored, and the mapping leads to the correct results.

In comparison with the approach of Pittel et al., the boson-fermion and the baryon mapping appear to be more systematic. The most important feature of the two latter methods is the generation of an exact image space and of extended image operators which, when acting on kets, leave this space invariant. This property is preserved when a truncation is performed, provided that the collective fermion space is invariant under the original fermionic operators. The construction of the collective extended images, however, can be a rather lengthy and complicated procedure. In this respect, the technique developed by Pittel et al. seems to be superior because it is easier to handle. For both approaches, in spite of their success in the case of the simple quark shell model, the applicability to more general problems remains to be shown.

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