Boson-fermion and baryon mapping: Construction of collective subspaces

I. Theory

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Abstract

Recently, the mathematical formalism of the Dyson boson mapping has been extended to a system of 3n fermions, leading to the boson-fermion and the baryon mapping. In the present paper, the case of a restriction to a subset of three-fermion quantum numbers, the collective indices, is discussed. A theory is developed for the representation of fermionic states and operators in a truncated ideal space where only collective boson-fermion pairs or collective ideal baryons are allowed. An exact reproduction of physical properties is proved to be possible provided that the original fermionic problem can be solved in a subspace where all three-fermion subsystems carry collective indices. Examples of simple applications are presented in the two subsequent papers of this series.

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I. Introduction

In a recent article,[1] the well-known formalism of the nonunitary Dyson boson mapping[2,3] has been extended to multifermion systems with three-particle substructures as occurring in the quark model of baryonic matter. The new approach is designed for the mathematical treatment of baryon systems on the basis of an effective interaction between quarks. The final aims are the calculation of the ground state and of low-lying excited states and the construction of simplified phenomenological models for the interaction of the baryons. Two different mathematical techniques have been presented.[1,4] The first method, the boson-fermion mapping, starts from a quark-diquark ansatz and leads to a description of a 3n-quark system in terms of n bosons and n so-called ideal fermions, where each boson represents a quark pair and each fermion stands for a single quark. The second alternative, the baryon mapping, is a direct step from states with 3n quarks to n-particle states. The particles created in this way show fermionic behaviour and are termed ideal baryons because they carry the quantum numbers of quark triplets.

Both methods are based on a nonunitary transformation of the original multifermion states into the ideal space, i. e., the space spanned by the many-particle states of the newly-introduced particles. At this stage, the underlying formalism is completely general and does not depend on the structure of the actual physical interaction. The crucial step where physics becomes relevant consists in a truncation of the ideal space. In the states of the resulting subspace, the quantum numbers of the ideal baryons or boson-fermion pairs are restricted to those of real bound three-quark or quark-diquark systems, respectively. The di- and triquark quantum numbers selected in this way will be termed *collective*, in accordance with the current terminology used in boson mapping theory (see, e. g., Ref. 5). As usual, the truncated space is referred to as the collective subspace of the ideal space.

The ultimate aims of the truncation procedure are a simplification of the original many-body problem and an approximate description of the multiquark states at lowest energy. For a good reproduction of physical properties, the structure of image operators acting in the truncated ideal space is essential. As in the case of the Dyson boson mapping,[6-13] the representation of an operator in the ideal space is in general not uniquely defined. As a rule, several different variants exist, leading to identical physical results[1,4] as long as no approximations are made. If a truncation is performed, however, the choice of a suitable variant determines whether a good approximation is obtained. The same is true also for the Dyson boson mapping.[9,13] There, provided that certain algebraic conditions are fulfilled,[5,14] it is possible even to derive *exact* results in terms of purely collective bosons.

The present paper is a first attempt to develop a similar theory for the boson-fermion and the baryon mapping. The mathematical foundations of the truncation procedure and the prerequisites for a representation of physical operators and states in the truncated ideal space are analyzed. An exact reproduction of physical properties is found to be possible if the original fermionic problem can be solved in a truncated multifermion space where the quantum numbers of the three-fermion subsystems are allowed to take only collective values. In the second paper of this series, the formalism developed here will be illustrated by the example of a simple physical model where the collective triquarks are nucleons. The more general case of baryons, i. e., of triquarks which are colourfree, and its treatment with the baryon mapping will be discussed in the third paper.

The present paper is organized as follows: Section II contains a short review of some foundations presented in Ref. 1. For details concerning the properties of the boson-fermion and the baryon mapping, the reader is referred to Refs. 1 or 4. In Sec. III, a new notation is introduced both for the states of two- and three-fermion systems and for the states of bosons, boson-fermion pairs, and ideal baryons. The subject of Sec. IV is the concept of collective subspaces which are obtained by means of a restriction to collective three-fermion quantum numbers. The questions concerning the existence and the properties of collective image operators are treated in Sec. V. The article ends with a short summary in Sec. VI.

II. Basic definitions

Let a^+_α, a_α with α = 1,...,m be the creation and destruction operators of fermions (here quarks) possessing m single-particle states. Both the boson-fermion and the baryon mapping are defined on the space $\mathcal{F}$ spanned by all states

$$a^+_{\alpha_n} a^+_{\beta_n} a^+_{\gamma_n} \ldots a^+_{\alpha_1} a^+_{\beta_1} a^+_{\gamma_1} |0\rangle \qquad (2.1)$$

with $n \in \mathbb{N}_0$, $3n \leq m$, and α_k, β_k, γ_k = 1,...,m for k = 1,...,n. As usual, $|0\rangle$ represents the vacuum state. $\mathcal{F}$ is mapped into the *ideal space* $\mathcal{J}$ consisting of all linear combinations of the many-particle states of bosons and ideal fermions or ideal baryons, respectively. The image of $\mathcal{F}$ is a subspace of $\mathcal{J}$, the *physical subspace* $\mathcal{P}$, where the Pauli principle is preserved. The elements of $\mathcal{P}$ are termed physical states. The total space $\mathcal{J}$ is the direct sum of $\mathcal{P}$ and its orthogonal complement, the *unphysical subspace* $\mathcal{P}^\perp$. The notation $|\ \rangle$ is used to denote kets in $\mathcal{F}$, whereas kets in $\mathcal{J}$ are written as $|\ )$. In particular, the symbol $|0)$ stands for the vacuum state of $\mathcal{J}$.

In the case of the boson-fermion mapping, boson creation and annihilation operators $b^+_{\alpha\beta}$, $b_{\alpha\beta}$ with α, β = 1,...,m are defined to be antisymmetric in their indices α and β and to satisfy the relation

$$[b_{\alpha\beta}, b^+_{\varkappa\lambda}] = \delta_{\alpha\varkappa}\delta_{\beta\lambda} - \delta_{\beta\varkappa}\delta_{\alpha\lambda} \ . \qquad (2.2)$$

The remaining commutators vanish. The ideal fermions are represented by the operators c^+_α, c_α with α = 1,...,m. These obey ordinary fermionic anti-commutation relations and commute with all boson operators. The ideal space $\mathcal{J}$ of the boson-fermion mapping is spanned by all states containing as many bosons as ideal fermions.

For the baryon mapping, ideal baryon creation and destruction operators $b^+_{\alpha\beta\gamma}$, $b_{\alpha\beta\gamma}$ with α, β, γ = 1,...,m are introduced. By definition, they are totally antisymmetric in their indices and fulfill the requirement[1,4]

$$\{b_{\alpha\beta\gamma}, b^+_{\kappa\lambda\mu}\} = \delta_{\alpha\kappa}\delta_{\beta\lambda}\delta_{\gamma\mu} + \delta_{\beta\kappa}\delta_{\gamma\lambda}\delta_{\alpha\mu} + \delta_{\gamma\kappa}\delta_{\alpha\lambda}\delta_{\beta\mu} - \delta_{\beta\kappa}\delta_{\alpha\lambda}\delta_{\gamma\mu} - \delta_{\gamma\kappa}\delta_{\beta\lambda}\delta_{\alpha\mu} - \delta_{\alpha\kappa}\delta_{\gamma\lambda}\delta_{\beta\mu} . \qquad (2.3)$$

The remaining anticommutators are zero. In Sec. III.B, it will become obvious that the particles defined in this way have fermionic properties. The ideal space corresponding to the baryon mapping is the Fock space of the ideal baryons.

In both cases, the physical subspace $\mathcal{P}$ is generated by the vacuum state $|0)$ and all elements of $\mathcal{J}$ which are totally antisymmetric with respect to interchanges of their fermion indices. The orthogonal projection operator in $\mathcal{J}$ onto the physical subspace $\mathcal{P}$ is denoted by $\hat{P}$. For an arbitrary linear operator $\hat{A} : \mathcal{F} \longrightarrow \mathcal{F}$, the image $\hat{A}_{\mathcal{P}}$ constructed with the boson-fermion or the baryon mapping can be expressed in the form[1,4]

$$\hat{A}_{\mathcal{P}} = \hat{A}_{\mathcal{J}} \; \hat{P} \; , \qquad (2.4)$$

where $\hat{A}_{\mathcal{J}}$ is a polynomial of creation and annihilation operators acting in $\mathcal{J}$. Since $\hat{A}_{\mathcal{P}}$ has the property[1,4]

$$\hat{A}_{\mathcal{P}} = \hat{A}_{\mathcal{P}} \; \hat{P} = \hat{P} \; \hat{A}_{\mathcal{P}} = \hat{P} \; \hat{A}_{\mathcal{P}} \; \hat{P} \; , \qquad (2.5)$$

it follows from Eq. (2.4) that $\hat{A}_{\mathcal{J}}$ leaves the space of physical *ket* states invariant, i. e., fulfills the relation

$$\hat{A}_{\mathcal{J}}\ \hat{P} = \hat{P}\ \hat{A}_{\mathcal{J}}\ \hat{P}\ . \tag{2.6}$$

(In general, due to the nonunitarity of the two mappings, the analogous statement for *bra* vectors does *not* hold.[1,4]) As a consequence of Eq. (2.6), it is possible to work with the simpler operator $\hat{A}_{\mathcal{J}}$ instead of $\hat{A}_{\mathcal{P}}$ without influencing the results obtained for physical ket states. It should, however, be noted that $\hat{A}_{\mathcal{J}}$, contrary to $\hat{A}_{\mathcal{P}}$, may produce nonvanishing results also when applied to *un*physical states. Therefore, $\hat{A}_{\mathcal{J}}$ is termed the *extended* image[1,4] of $\hat{A}$, as distinct from the *exact* image $\hat{A}_{\mathcal{P}}$. As mentioned in the Introduction, Eq. (2.6) does not determine $\hat{A}_{\mathcal{J}}$ uniquely but generally allows the existence of several variants agreeing only in their effect on physical ket states.

III. Coupling of fermion indices

As a preparation for the introduction of collective di- and trifermion indices, the coupling of three fermion indices by means of a linear transformation is treated in this section. The aim is to characterize the states of a three-fermion system (trifermion) by a single index describing the properties of the system as a whole. Two different approaches are possible. The first method, the *difermion-fermion coupling*, starts with the formation of *two*-fermion systems (difermions) which, in a second step, are coupled to a third fermion. This procedure is adapted to the structure of the boson-fermion mapping. In connection with the baryon mapping, the second approach is to be used. There, the coupling of the three fermions is achieved in a *single* step, and the corresponding transformation will be referred to as *trifermion coupling*. The two different types of coupling and their application to the boson-fermion and the baryon mapping, respectively, are described below.

A. Difermion-fermion coupling

In the following, small latin letters stand for *two*-fermion states, whereas capital latin letters denote states originating from the coupling of *three* fermion indices.

The annihilation and creation operators of a difermion in the state r are constructed with the aid of a linear and unitary transformation acting on a pair of fermion indices:[15,16]

$$(aa)_r \quad := 1/2 \sum_{\alpha\beta} C^{r*}_{\alpha\beta} \, a_\beta a_\alpha \ , \tag{3.1}$$

$$(a^+a^+)_r := (aa)_r^+ \; . \tag{3.2}$$

The transformation coefficients $C^r_{\alpha\beta}$ appearing in Eq. (3.1) are defined to be antisymmetric in their indices α and β and to satisfy the orthogonality and completeness relations[16]

$$1/2 \sum_{\alpha\beta} C^{r*}_{\alpha\beta} \, C^s_{\alpha\beta} = \delta_{rs} \; , \tag{3.3}$$

$$\sum_r C^r_{\alpha\beta} \, C^{r*}_{\varkappa\lambda} = \delta_{\alpha\varkappa}\delta_{\beta\lambda} - \delta_{\beta\varkappa}\delta_{\alpha\lambda} \; . \tag{3.4}$$

In the next step, a second linear transformation is performed. It couples difermion with single-fermion indices. For a three-fermion system in a state R constructed in this way, annihilation and creation operators can be introduced as follows:

$$(aaa)_R := \sum_r \sum_\rho D^{R*}_{r\rho} \; a_\rho (aa)_r \tag{3.5}$$

$$= 1/2 \sum_r \sum_{\alpha\beta\rho} C^{r*}_{\alpha\beta} \, D^{R*}_{r\rho} \; a_\rho a_\beta a_\alpha \; , \tag{3.6}$$

$$(a^+a^+a^+)_R := (aaa)_R^+ \; . \tag{3.7}$$

Here, the coefficients of the second transformation are denoted by $D^R_{r\rho}$. For simplicity, they are again chosen to fulfill the requirements of orthogonality and completeness, i. e.,

$$\sum_r \sum_\rho D^{R*}_{r\rho} \, D^S_{r\rho} = \delta_{RS} \; , \tag{3.8}$$

$$\sum_R D^R_{r\rho} \, D^{R*}_{s\sigma} = \delta_{rs}\delta_{\rho\sigma} \; . \tag{3.9}$$

The completeness relation (3.9) indicates that no restrictions are imposed on the coupling of difermion and single-fermion indices. This means that the coupling is not necessarily totally antisymmetric in all *three* of the original fermion indices but that combinations of mixed symmetry where only the *di*-fermion part is antisymmetrized are possible, too. In the trifermion operators given by Eq. (3.6), such contributions are of course zero. As a consequence, the maximum number of linearly independent three-fermion states is smaller than the number of different values of R obtainable by means of the transformation. The *full* set of indices R is therefore overcomplete and inappropriate for the classification of trifermion states. This is the price to be paid for the simple structure of the second transformation. In practical applications, however, one can easily overcome this disadvantage by working with a collective subset of indices which is chosen in such a way that the corresponding trifermion states are linearly independent (see Sec. IV.A).

For the application of the boson-fermion mapping, it is advisable to introduce new boson indices constructed in a similar manner as above. In accordance with the common practice in boson mapping theory,[15,16] the annihilation operator of a boson in the state r is defined by

$$b_r := 1/2 \sum_{\alpha\beta} C^{r*}_{\alpha\beta}\, b_{\alpha\beta}\ . \tag{3.10}$$

The inverse relation reads:

$$b_{\alpha\beta} = \sum_r C^r_{\alpha\beta}\, b_r\ . \tag{3.11}$$

The new boson operators obey the simple commutation rules

$$\left.\begin{aligned}&[b_r, b_s^+] = \delta_{rs}\ ,\\ &[b_r, b_s] = [b_r^+, b_s^+] = 0\ .\end{aligned}\right\} \tag{3.12}$$

In Refs. 1 and 4, it has been shown that the nonunitary boson-fermion mapping of the operators $a_\rho a_\beta a_\alpha$ and $a_\alpha^+ a_\beta^+ a_\rho^+$ leads to the results

$$(a_\rho a_\beta a_\alpha)_{\mathcal{P}} = (a_\rho a_\beta a_\alpha)_{\mathcal{J}}\ \hat{P} = b_{\alpha\beta} c_\rho\ \hat{P} \tag{3.13}$$

and

$$(a_\alpha^+ a_\beta^+ a_\rho^+)_{\mathcal{P}} = (a_\alpha^+ a_\beta^+ a_\rho^+)_{\mathcal{J}}\ \hat{P} = (Z_{\alpha\beta\rho}^+ + Z_{\beta\rho\alpha}^+ + Z_{\rho\alpha\beta}^+)\ \hat{P} \tag{3.14}$$

with

$$\begin{aligned} &Z_{\alpha\beta\rho}^+ \\ &:= \quad b_{\alpha\beta}^+ c_\rho^+ + \sum_{\varkappa} b_{\alpha\varkappa}^+ c_\beta^+ c_\rho^+ c_\varkappa + \sum_{\varkappa\lambda} b_{\alpha\lambda}^+ b_{\beta\varkappa}^+ c_\rho^+ b_{\varkappa\lambda} \\ &\quad + \Big(\sum_{\varkappa\lambda} b_{\alpha\beta}^+ b_{\rho\varkappa}^+ c_\lambda^+ b_{\varkappa\lambda} \\ &\qquad + 1/6 \sum_{\varkappa\lambda} b_{\varkappa\lambda}^+ c_\alpha^+ c_\beta^+ c_\rho^+ c_\lambda c_\varkappa \\ &\qquad + 1/3 \sum_{\substack{\varkappa\lambda\\ \mu\nu}} b_{\alpha\varkappa}^+ b_{\beta\lambda}^+ b_{\rho\mu}^+ c_\nu^+ b_{\varkappa\lambda} b_{\nu\mu} \Big) \frac{1}{\hat{N}+1} \end{aligned} \tag{3.15}$$

(see Eqs. (4.27a), (4.31), and (4.16) of Ref. 1). Here, $\hat{N}$ is a particle number operator counting the bosons or, equivalently, the ideal fermions in the states of $\mathcal{J}$, and the $\hat{N}$-dependent fraction has to be replaced by its eigenvalue when acting on a state with a definite particle number.

As a consequence of Eq. (3.13), the extended image of the trifermion operator $(aaa)_R$ given by Eqs. (3.5) - (3.6) can be chosen to be

$$\big((aaa)_R\big)_{\mathcal{J}} := (bc)_R \tag{3.16}$$

with the definition

$$(bc)_R := 1/2 \sum_r \sum_{\alpha\beta\rho} C^{r*}_{\alpha\beta}\, D^{R*}_{r\rho}\, b_{\alpha\beta}c_\rho \tag{3.17}$$

$$= \sum_r \sum_\rho D^{R*}_{r\rho}\, b_r c_\rho \tag{3.18}$$

for arbitrary values of R. Contrary to the fermionic operator $(aaa)_R$ in Eq. (3.6), the expression (3.17) is in general *not* completely antisymmetrized with respect to interchanges of the indices α, β, ρ. From the *exact* image $\big((aaa)_R\big)_{\mathcal{P}} = \big((aaa)_R\big)_{\mathcal{J}}\, \hat{P}$, however, all terms possessing no fermionic origin are excluded by the presence of $\hat{P}$ because the product $b_{\alpha\beta}c_\rho\, \hat{P}$ is totally antisymmetric in its three indices.[1,4]

One can invert the relation (3.17) by using the completeness property of the coupling coefficients (see Eqs. (3.4) and (3.9)). The result reads:

$$b_{\alpha\beta}c_\rho = \sum_R \sum_r C^r_{\alpha\beta}\, D^R_{r\rho}\, (bc)_R\ . \tag{3.19}$$

For creation operators, the coupling of a boson-fermion pair in the state R is defined by

$$(b^+c^+)_R := (bc)^+_R\ . \tag{3.20}$$

However, due to the complicated structure of Eqs. (3.14) - (3.15), this simple expression is *not* an extended image of $(a^+a^+a^+)_R$:[1,4]

$$\left((a^+a^+a^+)_R\right)_{\mathcal{J}} \neq (b^+c^+)_R \ . \tag{3.21}$$

B. Trifermion coupling

As in Sec. III.A, three-fermion states will be denoted by capital latin letters. However, the states are now constructed in a different way and are therefore generally not identical with those obtained by means of the difermion-fermion coupling. The new trifermion destruction and creation operators belonging to the state R have the structure

$$(aaa)_R \quad := 1/6 \sum_{\alpha\beta\gamma} C^{R*}_{\alpha\beta\gamma}\, a_\gamma a_\beta a_\alpha \ , \tag{3.22}$$

$$(a^+a^+a^+)_R := (aaa)^+_R \ , \tag{3.23}$$

where the transformation coefficients $C^R_{\alpha\beta\gamma}$ are required to be totally antisymmetric in α, β, γ and to possess the following orthogonality and completeness properties:[4]

$$1/6 \sum_{\alpha\beta\gamma} C^{R*}_{\alpha\beta\gamma}\, C^S_{\alpha\beta\gamma} = \delta_{RS} \ , \tag{3.24}$$

$$\begin{aligned}\sum_R C^R_{\alpha\beta\gamma}\, C^{R*}_{\varkappa\lambda\mu} = \ & \delta_{\alpha\varkappa}\delta_{\beta\lambda}\delta_{\gamma\mu} + \delta_{\beta\varkappa}\delta_{\gamma\lambda}\delta_{\alpha\mu} + \delta_{\gamma\varkappa}\delta_{\alpha\lambda}\delta_{\beta\mu} \\ & - \delta_{\beta\varkappa}\delta_{\alpha\lambda}\delta_{\gamma\mu} - \delta_{\gamma\varkappa}\delta_{\beta\lambda}\delta_{\alpha\mu} - \delta_{\alpha\varkappa}\delta_{\gamma\lambda}\delta_{\beta\mu} \ .\end{aligned} \tag{3.25}$$

(Here, contrary to the case of the difermion-fermion coupling, the total antisymmetry of the coefficients automatically ensures the linear independence of three-fermion states characterized by different values of R.)

In order to facilitate the handling of the baryon mapping, the same formalism is applied to the ideal baryon operators, yielding the result

$$b_R := 1/6 \sum_{\alpha\beta\gamma} C^{R*}_{\alpha\beta\gamma}\, b_{\alpha\beta\gamma} \tag{3.26}$$

for the annihilation operator of an ideal baryon in the state R. The inverse relation is given by

$$b_{\alpha\beta\gamma} = \sum_R C^R_{\alpha\beta\gamma}\, b_R \ . \tag{3.27}$$

With the aid of this formula, the anticommutation rules of the ideal baryons get the well-known fermionic structure

$$\left.\begin{aligned} &\{b_R,\ b_S^+\} = \delta_{RS}\ , \\ &\{b_R,\ b_S\} = \{b_R^+,\ b_S^+\} = 0\ , \end{aligned}\right\} \tag{3.28}$$

as already mentioned in Sec. II.

The baryon images of the operators $a_\gamma a_\beta a_\alpha$ and $a^+_\alpha a^+_\beta a^+_\gamma$ have been calculated in Refs. 1 and 4. The results are

$$(a_\gamma a_\beta a_\alpha)_{\mathcal{P}} = (a_\gamma a_\beta a_\alpha)_{\mathcal{J}}\ \hat{P} = b_{\alpha\beta\gamma}\ \hat{P} \tag{3.29}$$

and

$$(a^+_\alpha a^+_\beta a^+_\gamma)_{\mathcal{P}} = (a^+_\alpha a^+_\beta a^+_\gamma)_{\mathcal{J}}\ \hat{P}$$

$$= \Big(\ b^+_{\alpha\beta\gamma}$$

$$+ 1/2 \sum_{\varkappa\lambda\mu} (b^+_{\varkappa\lambda\alpha} b^+_{\beta\gamma\mu} + b^+_{\varkappa\lambda\beta} b^+_{\gamma\alpha\mu} + b^+_{\varkappa\lambda\gamma} b^+_{\alpha\beta\mu})\ b_{\varkappa\lambda\mu}$$

$$+ 1/8 \sum_{\substack{\varkappa\mu\rho \\ \lambda\nu\sigma}} b^+_{\varkappa\lambda\alpha} b^+_{\mu\nu\beta} b^+_{\rho\sigma\gamma} b_{\varkappa\mu\rho} b_{\lambda\nu\sigma}\ \Big)\ \hat{P} \tag{3.30}$$

(see Eqs. (5.23), (5.21a), and (5.15) of Ref. 1). They reflect the nonunitarity of the mapping.[1,4] Using Eq. (3.29), one immediately derives the expression

$$\big((aaa)_R\big)_{\mathcal{J}} := b_R \tag{3.31}$$

for the extended image of $(aaa)_R$. From Eq. (3.30), it follows that an equally simple relation for the creation operators does not exist, i. e.,

$$\big((a^+a^+a^+)_R\big)_{\mathcal{J}} \neq b^+_R\ . \tag{3.32}$$

IV. The concept of collective subspaces

A. Collective indices

The sets of two- and three-fermion indices r and R introduced in Sec. III are now reduced to subsets consisting of collective indices. Throughout this paper, the selected indices will be marked by a prime. Accordingly, r' is a collective index of a difermion or a boson, whereas R' stands for a collective state of a trifermion, a boson-fermion pair, or an ideal baryon.

In the case of the boson-fermion mapping, the choice of collective indices is subject to two additional conditions. First, as already mentioned in Sec. III.A, the collective three-fermion states are chosen to be linearly independent. In addition, the collective di- and trifermion indices are related through the requirement

$$D^{R'}_{r\rho} = 0 \qquad \text{for } \textit{non}\text{collective indices } r, \tag{4.1}$$

so that the collective trifermion operator $(aaa)_{R'}$ given by Eq. (3.5) can be expressed in the form

$$(aaa)_{R'} = \sum_{r'} \sum_{\rho} D^{R'*}_{r'\rho} \, a_\rho (aa)_{r'} \tag{4.2}$$

with purely collective difermion quantum numbers. Consequently, the corresponding image $(bc)_{R'}$ defined as in Eq. (3.18) contains only collective boson operators:

$$(bc)_{R'} = \sum_{r'} \sum_{\rho} D^{R'*}_{r'\rho} \, b_{r'} c_\rho \ . \tag{4.3}$$

B. Collective subspaces

The definitions presented in this subsection are kept completely general so as to hold for both the boson-fermion and the baryon mapping. As an abbreviation, the notation

$$(aaa)_R =: T_R \tag{4.4}$$

is adopted. Here, the difermion-fermion coupling is used in connection with the boson-fermion mapping, whereas the trifermion coupling belongs to the baryon mapping.

The spaces $\mathcal{F}$, $\mathcal{P}$, and $\mathcal{I}$ are defined as in Sec. II. In the following, several different collective subspaces will be introduced. The underlying concept is analogous to a scheme developed by Kim and Vincent[5] for the Dyson boson mapping. The nomenclature is chosen in such a way that subspaces of $\mathcal{F}$, $\mathcal{P}$, and $\mathcal{I}$ are labeled by the respective symbols with additional lower indices. In particular, the symbol $\mathcal{I}$ with a subscript indicates that the corresponding subspace of $\mathcal{I}$ is in general *not* contained in $\mathcal{P}$.

Let $\mathcal{F}_C$ be the *collective subspace of* $\mathcal{F}$ or, for short, the *collective fermion space.* $\mathcal{F}_C$ is spanned by the states of the form

$$T^{+}_{R'_n} \dots T^{+}_{R'_1}|0\rangle \tag{4.5}$$

with all possible integer values of n and arbitrary collective indices $R'_1,\dots,R'_n$.

Another important space is $\mathcal{I}_C$, the *collective subspace of* $\mathcal{I}$ or *collective ideal space.* It is spanned by the states possessing the structure

$$(T_{R'_n})^{+}_{\mathcal{I}} \dots (T_{R'_1})^{+}_{\mathcal{I}} \, |0) \ , \tag{4.6}$$

where, again, n and the collective indices take on arbitrary values. Explicitly, the operator $(T_{R'})^{+}_{\mathcal{J}}$ is given by

$$(T_{R'})^{+}_{\mathcal{J}} = (b^{+}c^{+})_{R'} \qquad \text{for the boson-fermion mapping} \tag{4.7}$$

and

$$(T_{R'})^{+}_{\mathcal{J}} = b^{+}_{R'} \qquad \text{for the baryon mapping.} \tag{4.8}$$

The relation between $\mathcal{J}$ and $\mathcal{J}_C$ is expressed by the equation

$$\mathcal{J}_C = \hat{C}\ \mathcal{J}\ , \tag{4.9}$$

where $\hat{C}$ is the orthogonal projection operator in $\mathcal{J}$ onto the collective subspace $\mathcal{J}_C$.

The image vectors corresponding to the states (4.5) of $\mathcal{F}_C$,

$$(T^{+}_{R'_n})_{\mathcal{P}} \ldots (T^{+}_{R'_1})_{\mathcal{P}}\ |0) = (T^{+}_{R'_n})_{\mathcal{J}}\ \hat{P} \ldots (T^{+}_{R'_1})_{\mathcal{J}}\ \hat{P}\ |0)\ , \tag{4.10}$$

span the subspace $\mathcal{P}_C$ of $\mathcal{P}$. $\mathcal{P}_C$ is the image of $\mathcal{F}_C$ in $\mathcal{P}$. By repeated use of Eq. (2.6), the states (4.10) are brought into the simple form

$$(T^{+}_{R'_n})_{\mathcal{J}} \ldots (T^{+}_{R'_1})_{\mathcal{J}}\ |0)\ . \tag{4.11}$$

It should be kept in mind that

$$(T^{+}_{R'})_{\mathcal{J}} \neq (T_{R'})^{+}_{\mathcal{J}}\ , \tag{4.12}$$

as already pointed out in Sec. III. The operator $(T^{\dagger}_{R'})_{\mathcal{I}}$ appearing in the states of $\mathcal{P}_C$ shows a far more complicated structure than $(T_{R'})^{\dagger}_{\mathcal{I}}$ given by Eq. (4.7) or (4.8). Contrary to the latter, $(T^{\dagger}_{R'})_{\mathcal{I}}$ may contain also *non*collective operators of bosons or of ideal baryons. This means that the space $\mathcal{P}_C$ is in general not a subspace of $\mathcal{I}_C$. According to the theory presented in Refs. 1 and 4 (in Ref. 1, see Eqs. (4.18) and (4.24a) as well as Eqs. (5.17) and (5.21a)), $\mathcal{P}_C$ can be written as the projection of $\mathcal{I}_C$ into $\mathcal{P}$, i. e.,

$$\mathcal{P}_C = \hat{P}\ \mathcal{I}_C = \hat{P}\ \hat{C}\ \mathcal{I}\ , \tag{4.13}$$

and is therefore termed the *physical collective subspace* of $\mathcal{I}$. The orthogonal projection operator in $\mathcal{I}$ onto $\mathcal{P}_C$ will be denoted by $\hat{P}_C$. Note that it is not identical with the product $\hat{P}\ \hat{C}$, which is not a projector because $\hat{P}$ and $\hat{C}$ do not commute. Since $\mathcal{P}_C$ is a subspace of $\mathcal{P}$, the operator $\hat{P}_C$ possesses the property

$$\hat{P}_C = \hat{P}\ \hat{P}_C = \hat{P}_C\ \hat{P} = \hat{P}\ \hat{P}_C\ \hat{P}\ . \tag{4.14}$$

In order to represent the physics of $\mathcal{F}_C$ in the truncated ideal space $\mathcal{I}_C$, one has to find a collective analogue of the physical subspace $\mathcal{P}$, i. e., a subspace of $\mathcal{I}_C$ taking the role of $\mathcal{P}$ after the truncation. In Sec. V, the space fulfilling this requirement will be demonstrated to be the subspace $\mathcal{I}_{CPC}$ of $\mathcal{I}_C$, defined by

$$\mathcal{I}_{CPC} := \hat{C}\ \mathcal{P}_C = \hat{C}\ \hat{P}\ \mathcal{I}_C = \hat{C}\ \hat{P}\ \hat{C}\ \mathcal{I}\ . \tag{4.15}$$

Since noncollective operators of bosons or of ideal baryons are excluded from its states, $\mathcal{I}_{CPC}$ is in general not a subspace of $\mathcal{P}$. Here, $\mathcal{I}_{CPC}$ is called the *model space* (cf. the nomenclature used by Kim and Vincent in Ref. 5).

The relationship between the various spaces introduced above is illustrated by the diagram in Fig. 1. For further reference, the definitions of the different subspaces of the ideal space $\mathcal{I}$ are summarized in Table I.

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\text{Mapping}"] \arrow[d, dashed, "\substack{\text{Restriction to}\\ \text{collective}\\ \text{trifermions}}"'] & \mathcal{P} \arrow[r, phantom, "\subset"] \arrow[d, dashed] & \mathcal{I} \arrow[dd, dotted, "\hat{C}"'] \\
\mathcal{F}_C \arrow[r] & \mathcal{P}_C \arrow[d, dotted, "\hat{C}"'] & \\
 & \mathcal{I}_{CPC} \arrow[r, phantom, "\subset"] & \mathcal{I}_C
\end{tikzcd}
$$

FIG. 1: Truncation scheme

TABLE I: Subspaces of the ideal space $\mathcal{I}$

Physical subspace:	$\mathcal{P}$	$=$	$\hat{P}\,\mathcal{I}$				
Collective subspace:	$\mathcal{I}_C$	$=$	$\hat{C}\,\mathcal{I}$				
Physical collective subspace:	$\mathcal{P}_C$	$=$	$\hat{P}\,\hat{C}\,\mathcal{I}$	$=$	$\hat{P}_C\,\mathcal{I}$	$\subset$	$\mathcal{P}$
Model space:	$\mathcal{I}_{CPC}$	$=$	$\hat{C}\,\hat{P}\,\hat{C}\,\mathcal{I}$	$=$	$\hat{C}\,\hat{P}_C\,\mathcal{I}$	$\subset$	$\mathcal{I}_C$

V. Collective image operators

A. General mathematical background

Let $\hat{A} : \mathcal{F} \longrightarrow \mathcal{F}$ be a linear operator possessing the property

$$\hat{A}\,|f_C\rangle \in \mathcal{F}_C \qquad \forall \;\; |f_C\rangle \in \mathcal{F}_C \; . \tag{5.1}$$

An analogous relation holds for the image $\hat{A}_{\mathcal{P}}$ derived by applying the boson-fermion or the baryon mapping. Since the transformation preserves all matrix elements,[1,4] $\hat{A}_{\mathcal{P}}$ leaves the image $\mathcal{P}_C$ of $\mathcal{F}_C$ invariant, i. e.,

$$\hat{A}_{\mathcal{P}}\,|p_C) \in \mathcal{P}_C \qquad \forall \;\; |p_C) \in \mathcal{P}_C \; . \tag{5.2}$$

For states of $\mathcal{F}_C$ with the general structure (4.5), the matrix elements of $\hat{A}$ before and after the transformation can be written as

$$\begin{aligned}
&\langle 0|T_{S_1'} \ldots T_{S_k'} \; \hat{A} \; T^{+}_{R_n'} \ldots T^{+}_{R_1'}|0\rangle \\
&= (0|\;(T_{S_1'})_{\mathcal{P}} \;\ldots\; (T_{S_k'})_{\mathcal{P}} \; \hat{A}_{\mathcal{P}} \; (T^{+}_{R_n'})_{\mathcal{P}} \;\ldots\; (T^{+}_{R_1'})_{\mathcal{P}} \; |0) \\
&= (0|\;(T_{S_1'})_{\mathcal{J}} \; \hat{P} \;\ldots\; (T_{S_k'})_{\mathcal{J}} \; \hat{P} \; \hat{A}_{\mathcal{P}} \; (T^{+}_{R_n'})_{\mathcal{J}} \; \hat{P} \;\ldots\; (T^{+}_{R_1'})_{\mathcal{J}} \; \hat{P} \; |0) \\
&= (0|\;(T_{S_1'})_{\mathcal{J}} \;\ldots\; (T_{S_k'})_{\mathcal{J}} \; \hat{A}_{\mathcal{P}} \; (T^{+}_{R_n'})_{\mathcal{J}} \;\ldots\; (T^{+}_{R_1'})_{\mathcal{J}} \; |0) \; .
\end{aligned} \tag{5.3}$$

In the last line, $\hat{P}$ has been shifted to the right by means of Eqs. (2.6) and (2.5) and then dropped. The bra vector is purely collective now, whereas the ket, expressed in the form (4.11), is an element of $\mathcal{P}_C$. Consequently, the operator $\hat{A}_{\mathcal{P}}$ may be replaced here by $\hat{C}\ \hat{A}_{\mathcal{P}}\ \hat{P}_C$ without affecting the matrix elements.

Since the physical collective subspace $\mathcal{P}_C$ is invariant under $\hat{A}_{\mathcal{P}}$ (see (5.2)), the operator $\hat{C}\ \hat{A}_{\mathcal{P}}\ \hat{P}_C$ has the property of mapping $\mathcal{P}_C$ into the model space $\hat{C}\ \mathcal{P}_C = \mathcal{I}_{CPC}$:

$$\left.\begin{aligned} \hat{C}\ \hat{A}_{\mathcal{P}}\ \hat{P}_C\ |p_C) &= \hat{C}\ \hat{A}_{\mathcal{P}}\ |p_C) \\ &\in \hat{C}\ \mathcal{P}_C = \mathcal{I}_{CPC} \qquad \forall\ \ |p_C) \in \mathcal{P}_C\ . \end{aligned}\right\} \tag{5.4}$$

As proved in the Appendix, the projection operator $\hat{C}$ defines a *one-to-one* correspondence between $\mathcal{P}_C$ and $\mathcal{I}_{CPC}$, i. e., the restricted mapping

$$\left.\begin{aligned} \hat{C}|\mathcal{P}_C :\ \ &\mathcal{P}_C \longrightarrow \hat{C}\ \mathcal{P}_C = \mathcal{I}_{CPC}\ , \\ &|p_C) \longmapsto \hat{C}\ |p_C)\ , \end{aligned}\right\} \tag{5.5}$$

is bijective. Therefore, the image $\hat{C}\ \hat{A}_{\mathcal{P}}\ |p_C)$ of the vector $|p_C)$ under $\hat{C}\ \hat{A}_{\mathcal{P}}\ \hat{P}_C$ can be as well interpreted as an image of $\hat{C}\ |p_C)$. This amounts to introducing an operator $\hat{A}'_{\mathcal{I}}$ whose action on $\mathcal{I}_{CPC}$ is determined by the relation

$$\hat{A}'_{\mathcal{I}}\ \hat{C}\ |p_C) := \hat{C}\ \hat{A}_{\mathcal{P}}\ |p_C) \qquad \forall\ \ \hat{C}\ |p_C) \in \mathcal{I}_{CPC}\ . \tag{5.6}$$

Since, due to (5.4), the right side of Eq. (5.6) is an element of $\mathcal{I}_{CPC}$, this definition implies that $\hat{A}'_{\mathcal{I}}$ maps the model space into itself:

$$\hat{A}'_{\mathcal{I}}\ \hat{C}\ |p_C) \in \mathcal{I}_{CPC} \qquad \forall\ \ \hat{C}\ |p_C) \in \mathcal{I}_{CPC}\ . \tag{5.7}$$

Moreover, it follows from the linearity of $\hat{A}$ and $\hat{A}_{\mathcal{P}}$ that $\hat{A}'_{\mathcal{I}}$ is linear on

$\mathcal{J}_{CPC}$. The defining relation (5.6) of $\hat{A}'_{\mathcal{J}}$ is equivalent to

$$\hat{C}\ \hat{A}_{\mathcal{P}}\ \hat{P}_C = \hat{A}'_{\mathcal{J}}\ \hat{C}\ \hat{P}_C\ . \tag{5.8}$$

The matrix element (5.3) can now be written as a matrix element of $\hat{A}'_{\mathcal{J}}$. First, as announced above, $\hat{C}\ \hat{A}_{\mathcal{P}}\ \hat{P}_C$ is substituted for $\hat{A}_{\mathcal{P}}$, yielding

$$\begin{aligned} &\langle 0| T_{S'_1} \ldots T_{S'_k}\ \hat{A}\ T^{+}_{R'_n} \ldots T^{+}_{R'_1} |0\rangle \\ &= (0|\ (T_{S'_1})_{\mathcal{J}}\ \ldots\ (T_{S'_k})_{\mathcal{J}}\ \hat{C}\ \hat{A}_{\mathcal{P}}\ \hat{P}_C\ (T^{+}_{R'_n})_{\mathcal{J}}\ \ldots\ (T^{+}_{R'_1})_{\mathcal{J}}\ |0)\ . \end{aligned} \tag{5.9}$$

By means of Eq. (5.8), one derives:

$$\begin{aligned} &\langle 0| T_{S'_1} \ldots T_{S'_k}\ \hat{A}\ T^{+}_{R'_n} \ldots T^{+}_{R'_1} |0\rangle \\ &= (0|\ (T_{S'_1})_{\mathcal{J}}\ \ldots\ (T_{S'_k})_{\mathcal{J}}\ \hat{A}'_{\mathcal{J}}\ \hat{C}\ (T^{+}_{R'_n})_{\mathcal{J}}\ \ldots\ (T^{+}_{R'_1})_{\mathcal{J}}\ |0)\ . \end{aligned} \tag{5.10}$$

The projection operator $\hat{P}_C$ originating from Eq. (5.8) has been omitted here because it would have been applied to a physical collective ket state. In the resulting matrix element, the operator $\hat{A}'_{\mathcal{J}}$ acts on a collective bra state and a ket state belonging to the model space $\mathcal{J}_{CPC}$. Since $\hat{A}'_{\mathcal{J}}$ maps $\mathcal{J}_{CPC}$ into itself and the definition (5.6) does not make any statement concerning the behaviour outside of $\mathcal{J}_{CPC}$, one can choose $\hat{A}'_{\mathcal{J}}$ in such a way that it contains no noncollective operators of bosons or of ideal baryons. From now on, $\hat{A}'_{\mathcal{J}}$ will be assumed to have this property. The original physical problem in $\mathcal{F}_C$ and its results are then exactly reproduced in the collective ideal space $\mathcal{J}_C$, where the subspace $\mathcal{J}_{CPC}$ takes the role of the image space. In this formalism, the invariance of $\mathcal{J}_{CPC}$ under $\hat{A}'_{\mathcal{J}}$, as expressed by (5.7), is the analogue of the invariance of $\mathcal{P}$ under the extended image $\hat{A}_{\mathcal{J}}$ of $\hat{A}$, i. e.,

$$\hat{A}_{\mathcal{J}}\ |p) \in \mathcal{P} \qquad \forall\ \ |p) \in \mathcal{P} \tag{5.11}$$

(see Eq. (2.6)), whereas (5.10) resembles the general relation

$$\begin{aligned} &\langle 0|T_{S_1}...T_{S_k}\ \hat{A}\ T^{+}_{R_n}...T^{+}_{R_1}|0\rangle \\ &= (0|\ (T_{S_1})_{\mathcal{J}}\ ...\ (T_{S_k})_{\mathcal{J}}\ \hat{A}_{\mathcal{J}}\ (T^{+}_{R_n})_{\mathcal{J}}\ ...\ (T^{+}_{R_1})_{\mathcal{J}}\ |0) \end{aligned} \tag{5.12}$$

for the matrix elements. (Equation (5.12) is obtained in the same way as (5.3) but with the additional substitution of $\hat{A}_{\mathcal{J}}\ \hat{P}$ for $\hat{A}_{\mathcal{P}}$.) In order to lay special emphasis on the analogy between $\hat{A}'_{\mathcal{J}}$ and $\hat{A}_{\mathcal{J}}$, the operator $\hat{A}'_{\mathcal{J}}$ will be termed the *collective extended image* of $\hat{A}$. Similar to $\hat{A}_{\mathcal{J}}$, $\hat{A}'_{\mathcal{J}}$ is not uniquely defined, for the relation (5.6) determines the effect of $\hat{A}'_{\mathcal{J}}$ on kets of $\mathcal{J}_{CPC}$ only. In general, several different variants may be possible, all satisfying the basic requirement given by Eq. (5.6) or (5.8).

In the second paper of this series, an example will be presented for the practical construction of collective extended image operators. The method described there is based on the relation (5.8). The same procedure will be applied also in the third paper.

B. Structure of the model space

In this subsection, the general formalism developed in Sec. V.A is applied to the special case of the operator

$\hat{A} = T^{+}_{R'}$ with a collective index R'.

(For the corresponding annihilation operator $T_{R'}$, the extended image is already purely collective, as seen from Eqs. (4.4), (3.16), and (3.31).) The collective extended image $(T^{+}_{R'})'_{\mathcal{J}}$ of $T^{+}_{R'}$ satisfies the relation (5.8), i. e.,

$$\hat{C}\ (T^{+}_{R'})_{\mathcal{P}}\ \hat{P}_C = (T^{+}_{R'})'_{\mathcal{J}}\ \hat{C}\ \hat{P}_C\ . \tag{5.13}$$

Since, as expressed by (5.2), the operator $(T^{+}_{R'})_{\mathcal{P}}$ leaves the space of physical collective kets invariant, this is equivalent to

$$\hat{C}\ \hat{P}_C\ (T^{+}_{R'})_{\mathcal{P}}\ \hat{P}_C = (T^{+}_{R'})'_{\mathcal{J}}\ \hat{C}\ \hat{P}_C\ . \tag{5.14}$$

The model space $\mathcal{J}_{CPC}$ is spanned by the states of the form

$$\hat{C}\ (T^{+}_{R'_n})_{\mathcal{P}}\ \ldots\ (T^{+}_{R'_1})_{\mathcal{P}}\ |0) = \hat{C}\ (T^{+}_{R'_n})_{\mathcal{J}}\ \ldots\ (T^{+}_{R'_1})_{\mathcal{J}}\ |0)\ , \tag{5.15}$$

where n and the collective indices $R'_1,\ldots,R'_n$ take all possible values. Equation (5.10) shows that these states play the role of image vectors belonging to the fermionic states (4.5). Therefore, the states (5.15) will be termed the *collective images* of (4.5). In Eq. (5.15), the projection operator $\hat{C}$ acts on a ket state with the structure (4.10). Since any such state belongs to $\mathcal{P}_C$, one can write:

$$\hat{C}\ (T^{+}_{R'_n})_{\mathcal{P}}\ (T^{+}_{R'_{n-1}})_{\mathcal{P}}\ \ldots\ (T^{+}_{R'_1})_{\mathcal{P}}\ |0)$$

$$= \hat{C}\ \hat{P}_C\ (T^{+}_{R'_n})_{\mathcal{P}}\ \hat{P}_C\ (T^{+}_{R'_{n-1}})_{\mathcal{P}}\ \hat{P}_C\ \ldots\ \hat{P}_C\ (T^{+}_{R'_1})_{\mathcal{P}}\ \hat{P}_C\ |0)\ . \tag{5.16}$$

Application of Eq. (5.14) to the operator $(T^+_{R'_n})_{\mathcal{P}}$ on the right side then gives:

$$\hat{C}\ (T^+_{R'_n})_{\mathcal{P}}\ (T^+_{R'_{n-1}})_{\mathcal{P}}\ \dots\ (T^+_{R'_1})_{\mathcal{P}}\ |0)$$

$$= (T^+_{R'_n})'_{\mathcal{J}}\ \hat{C}\ \hat{P}_C\ (T^+_{R'_{n-1}})_{\mathcal{P}}\ \hat{P}_C\ \dots\ \hat{P}_C\ (T^+_{R'_1})_{\mathcal{P}}\ \hat{P}_C\ |0)\ . \tag{5.17}$$

Now, the same procedure can be repeated with the remaining operators $(T^+_{R'_{n-1}})_{\mathcal{P}}$, ..., $(T^+_{R'_1})_{\mathcal{P}}$, leading to the final result

$$\hat{C}\ (T^+_{R'_n})_{\mathcal{P}}\ \dots\ (T^+_{R'_1})_{\mathcal{P}}\ |0) = (T^+_{R'_n})'_{\mathcal{J}}\ \dots\ (T^+_{R'_1})'_{\mathcal{J}}\ |0) \tag{5.18}$$

for the states of the model space. This equation is comparable to the general relation

$$(T^+_{R_n})_{\mathcal{P}}\ \dots\ (T^+_{R_1})_{\mathcal{P}}\ |0) = (T^+_{R_n})_{\mathcal{J}}\ \dots\ (T^+_{R_1})_{\mathcal{J}}\ |0)\ , \tag{5.19}$$

which holds for states of $\mathcal{P}$ and is derived by repeated application of Eqs. (2.4) and (2.6). Again, the analogy between the extended and the collective extended image becomes obvious.

VI. Summary and conclusions

The main results of this article can be summarized as follows: For any linear fermionic operator $\hat{A}$ leaving the collective subspace $\mathcal{F}_C$ of $\mathcal{F}$ invariant, the existence of a collective extended image $\hat{A}'_{\mathcal{J}}$ is automatically guaranteed. As a consequence, an exact description of physical properties in terms of collective boson-fermion systems or collective ideal baryons is possible. In this formalism, the space $\mathcal{F}_C$ is mapped onto the model space $\mathcal{J}_{CPC}$, where every collective trifermion operator $(a^+a^+a^+)_{R'}$ is represented by its collective extended image $\left((a^+a^+a^+)_{R'}\right)'_{\mathcal{J}}$. The mathematical formulas involved bear strong resemblance to the general case where no truncation is performed.

The central definition (5.8) of the collective extended image $\hat{A}'_{\mathcal{J}}$, however, is neither unique nor contains any further information on the actual mathematical structure of the operator. In the second paper of this series, the practical construction of collective extended images will be exemplified for the special case of a simple physical model. Useful formulas for the general case of a system of colourfree triquarks will be presented in the third paper.

Appendix:

Bijectivity of the mapping (5.5)

In order to prove that the mapping (5.5), given by

$$\left.\begin{aligned} \hat{C}|\mathcal{P}_C : \ \mathcal{P}_C &\longrightarrow \mathcal{I}_{CPC} \ , \\ |p_C) &\longmapsto \hat{C}\ |p_C) \ , \end{aligned}\right\} \qquad \text{(A.1)}$$

is bijective, it suffices to show that $\hat{C}|\mathcal{P}_C$ is injective, for the surjectivity is already implied in the definition of $\mathcal{I}_{CPC} = \hat{C}\ \mathcal{P}_C$. The injectivity is verified as follows:

Let $|p_{C1})$, $|p_{C2}) \in \mathcal{P}_C$ be two arbitrary physical collective states whose collective components are equal, i. e.,

$$\hat{C}\ |p_{C1}) = \hat{C}\ |p_{C2}) \ . \qquad \text{(A.2)}$$

As an abbreviation, the notation

$$|p_C) := |p_{C1}) - |p_{C2}) \qquad \text{(A.3)}$$

is introduced. Since $\mathcal{P}_C = \hat{P}\ \hat{C}\ \mathcal{I}$, there exists a state $|i) \in \mathcal{I}$ with the property

$$|p_C) = \hat{P}\ \hat{C}\ |i) \ . \qquad \text{(A.4)}$$

Equation (A.2) can now be written as

$$\hat{C}\ \hat{P}\ \hat{C}\ |i) = 0 \ , \qquad \text{(A.5)}$$

and it follows that

$$(i|\ \hat{C}\ \hat{P}\ \hat{C}\ |i) = 0\ . \tag{A.6}$$

Due to the idempotency and the self-adjointness of the projection operators, this matrix element is equal to

$$(i|\ \hat{C}\ \hat{P}^2\ \hat{C}\ |i) = (p_C|p_C) = 0\ . \tag{A.7}$$

Consequently, one arrives at the conclusion that

$$|p_C) = 0 \tag{A.8}$$

or, equivalently,

$$|p_{C1}) = |p_{C2})\ . \tag{A.9}$$

Together with Eq. (A.2), this means that the mapping (A.1) is injective, q. e. d.

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References

[1] J. Meyer, J. Math. Phys. **32**, 2142 (1991)

[2] D. Janssen, F. Dönau, S. Frauendorf, and R. V. Jolos, Nucl. Phys. **A 172**, 145 (1971)

[3] F. Dönau and D. Janssen, Nucl. Phys. **A 209**, 109 (1973)

[4] J. Meyer, *Boson-Fermion- und Baryonabbildung von Multiquarkzuständen*, doctoral thesis, University of Oldenburg, 1990

[5] G.-K. Kim and C. M. Vincent, Phys. Rev. **C 37**, 2176 (1988)

[6] H. B. Geyer and S. Y. Lee, Phys. Rev. **C 26**, 642 (1982)

[7] C.-T. Li, Phys. Rev. **C 29**, 2309 (1984)

[8] P. Park, Phys. Rev. **C 35**, 807 (1987)

[9] F. J. W. Hahne, H. B. Geyer, and C. A. Engelbrecht, Phys. Rev. **C 37**, 885 (1988)

[10] R. Kuchta, Phys. Rev. **C 38**, 1460 (1988)

[11] R. Kuchta, Nucl. Phys. **A 489**, 1 (1988)

[12] E. R. Marshalek, Phys. Rev. **C 38**, 2961 (1988)

[13] C. M. Vincent, L. J. Henry, and G.-K. Kim, Phys. Rev. **C 39**, 1583 (1989)

[14] R. Kuchta, Phys. Rev. **C 40**, 2930 (1989)

[15] P. Ring and P. Schuck, Phys. Rev. **C 16**, 801 (1977)

[16] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, New York, 1980)